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Squared Möbius Function for Half-Integral Matrices and Its Applications

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We define an analogue of the square of the usual Möbius function on the set of non-degenerate half-integral matrices. By using this function, we give a reasonable expression of the Koecher–Maaß Dirichlet series for a Siegel Modular form. In addition, we give another proof to a main result of (T. Ibukiyama and H. Saito, 1995, *Amer. J. Math.* **117**, 1097–1155) on the zeta functions for symmetric matrices.

INTRODUCTION

In this paper, on the set of non-degenerate half-integral symmetric matrices, we define an analogue of the square of the usual Möbius function and apply it to the zeta functions for Siegel modular forms and for quadratic forms.

Although our motivations for constructing such a function have been explained in [Ka], we here repeat one of them. Let f be a modular form of weight k belonging to the symplectic group $Sp_n(\mathbf{Z})$. Then $f(Z)$ has the following Fourier expansion,

$$f(Z)=\sum_{A} a_f(A)\, exp(2\pi i\, tr(AZ)),$$

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where A runs over all semi-positive definite half-integral matrices of degree n and tr denotes the trace of a matrix. We then define the Koecher–Maaß Dirichlet series $L(f, s)$ for f by

$$L(f, s) = \sum_{A} \frac{a_f(A)}{a(A, A)(\det A)^s},$$

where A runs over all $GL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree n, and $a(A, A)$ is the representation number of A by A itself. Furthermore, assume that f is a Hecke-eigenform, and let $\zeta^+(f, s)$ denote the standard L-function of f. The precise definition of this Dirichlet series will be given in Section 3. Now we are interested in a reasonable expression of $L(f, s)$ in terms of $\zeta^+(f, s)$. The case of $n = 1$ is very simple. In this case, f has a Fourier expansion of the form

$$f(z) = \sum_{m=0}^{\infty} a(m)\, exp(2\pi imz),$$

and $L(f, s)$ can be expressed as

$$L(f, s) = \frac{1}{2} \sum_{m=1}^{\infty} \frac{a(m)}{m^s}.$$

Thus by a simple calculation $L(f, s)$ can be expressed as

$$L(f, s) = \frac{1}{2}\, \zeta^+(f, 2s - k + 1) \times \sum_{m=1}^{\infty} \frac{\mu(m)^2\, B(2s - k + 1, m)}{m^s}\, a(m),$$

where μ is the Möbius function and $B(2s - k + 1, m) = \prod_{p \nmid m} (1 + p^{-2s+k-1})$. Here we remark that $\mu(m)^2 = 1$ or 0 according as m is square free or not. This expression looks trivial. This observation, however, plays an important role in our result. In fact, the above equality can be generalized to the higher dimensional case, and $L(f, s)$ can be expressed in a similar way in terms of the standard L-function and an analogue of μ^2, which is a main subject of the present paper.

This paper is organized as follows. In Section 1, on the set of non-degenerate half-integral symmetric matrices, we define an analogue σ of μ^2, which we call the squared Möbius function, and give a recursion formula for the representation numbers of half-integral matrices in terms of that function (cf. Theorem 1). After reviewing some preliminary results on the local theory of quadratic forms, we prove Theorem 1 in Section 2. Our squared Möbius function σ has several remarkable properties; firstly $\sigma(A)$ depends only on the genus of A, and secondly it has a relatively small support $\mathcal{K}_n(\mathbf{Z})$, which is an analogue of the set of square free integers. (The definition of $\mathcal{K}_n(\mathbf{Z})$ will be given in Section 1.) In Section 3, we relate the Koecher–Maaß Dirichlet series for a Siegel modular form to the standard L-function for it by using the squared Möbius function. To be more precise, we obtain the following expression of $L(f, s)$:

$$L(f, s) = \zeta^{+}(f, 2s-k+1) \sum_{\mathcal{G}(C_0)} G_f^*(C_0)\, R(C_0, s),$$

where $\mathcal{G}(C_0)$ runs over all genera of $\mathcal{K}_n(\mathbf{Z}) \cap \mathcal{H}_n(\mathbf{Z})^+$, $G_f^*(C_0)$ is *an weighted average sum* of the *primitive Fourier coefficients* of f over the genus of C_0, and $R(C_0, s)$ is a certain Dirichlet series explicitly expressed in terms of σ and the genus of C_0 (cf. Theorem 3.4). This is a weak generalization of a result of Böcherer [B2, Satz 1] which relates the Koecher–Maaß Dirichlet series to the standard L-function for a Siegel modular form of degree 2. In fact, by Theorem 3.4, we can easily derive his result. As for this, we will discuss it in a subsequent paper. In general, *primitive Fourier coefficients* are easier to treat than usual Fourier coefficients. Furthermore, in the above expression, we have only to treat $G_f^*(C_0)$ for $C_0 \in \mathcal{K}_n(\mathbf{Z}) \cap \mathcal{H}_n(\mathbf{Z})^+$. This enables us to deal with $L(f, s)$ very easily. In fact, by Theorem 3.4 we can obtain an explicit form of the Koecher–Maaß Dirichlet series for a special type of Siegel modular form, for an example, the Eisenstein series of Klingen type. As for this, in [Ka], the second author has announced some result which generalizes [B2, Satz 2], and we will discuss it more precisely in a subsequent paper [I-K2]. We note that our squared Möbius function also enables us to give a reasonable expression for the zeta function for half-integral matrices, and as a result we can give a short proof to the main result of the first author and Saito [I-S]. We discuss this topic in Section 4.

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Notation. As usual, by $\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$ we denote the ring of rational integers and the fields of rational, real and complex numbers, respectively. Furthermore, for a prime number p, we denote by $\mathbf{Z}_p$ and $\mathbf{Q}_p$ the ring of p-adic integers and the field of p-adic numbers, respectively.

For a subset S of a commutative ring R, put $S^{\square} = \{a^2; a \in S\}$. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m, n)-matrices with entries in R. Here we understand $M_{mn}(R)$ the set of the *empty matrix* if $m=0$ or $n=0$. In particular put $M_n(R)=M_{nn}(R)$. For an (m,n)-matrix X and an (m,m)-matrix A, we write $A[X]={}^tXAX$, where ${}^tX$ denotes the transpose of X. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset X mod $aM_{mn}(R)$. Put

$$GL_m(R)=\{A\in M_m(R);\ \det A\in R^*\},$$

where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. In particular, put $\Lambda_n=GL_n(\mathbf{Z})$ and $\Lambda_{np}=GL_n(\mathbf{Z}_p)$. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, for an integral domain R of characteristic different from 2, let $\mathscr{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathscr{H}_n(R)$ is the set of symmetric matrices of degree n whose (i,j)-components belong to R or $\frac{1}{2}R$ according as $i=j$ or not. We define the set $\mathscr{E}_n(R)$ of even-integral matrices over R by $\mathscr{E}_n(R)=2\mathscr{H}_n(R)$. For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of $S_n(\mathbf{R})$, we denote by S^+ the subset of S consisting of positive definite matrices. Let R' be a subring of R. Two symmetric matrices A and A' with entries in R are called equivalent over R' with each other and write $A \underset{R'}{\sim} A'$ if there is an element X of $GL_n(R')$ such that $A'=A[X]$. For square matrices X and Y we write $X\perp Y=\begin{pmatrix} X & O \\ O & Y\end{pmatrix}$. We also make the convention that $X\perp Y=X$ if Y is the empty matrix.

Let (M,q) be a quadratic module over a ring R with a quadratic form q in the sense of [Ki3], and b the associated symmetric bilinear form defined by $b(x,y)=q(x+y)-q(x)-q(y)$ for $x,y\in M$. In particular, if R is an integral domain of characteristic different from 2, we define a bilinear form B on M by $B(x,y)=\frac{1}{2}b(x,y)$. We define submodules $M^\perp$ and $\operatorname{Rad} M$ by

$$M^\perp=\{x\in M;\ b(x,y)=0 \text{ for any } y\in M\},$$

and

$$\operatorname{Rad} M=\{x\in M^\perp;\ q(x)=0 \text{ for any } x\in M\}.$$

We note that $M^\perp=\operatorname{Rad} M$ if the characteristic of R is not 2. We say that q is non-degenerate if $\operatorname{Rad} M=\{0\}$. For a half-integral matrix A over a ring R of degree n, let q_A be the quadratic form of $M_{n1}(R)$ over R defined by $q_A(\mathbf{x})=A[\mathbf{x}]$ for $\mathbf{x}\in M_{n1}(R)$. Then the quadratic space $M_A=(M_{n1}(R),q_A)$ is called the quadratic space associated with A.

Let F be a field of characteristic different from 2, and R a subring of F. Let (V,q) be a quadratic space over F, and L a quadratic R-lattice in V, that is, a finitely generated R-module such that $L \otimes_R F = V$. Then define the scale $s(L)$ and the norm $n(L)$ of L by

$$s(L) = \{B(x, y); x, y \in L\}, \qquad n(L) = \left\{\sum_i a_i q(x_i); a_i \in R, x_i \in L\right\}.$$

For a symmetric matrix A with entries in F, let $M_A = (M_{n1}(F), q_A)$ be the quadratic space associated with A as above. Then $M_{n1}(R)$ can be regarded as a quadratic R-lattice of M_A in a natural manner, which we write as L_A. We then put $s(A) = s(L_A)$ (resp. $n(A) = n(L_A)$), and call it the scale (resp. norm) of A.

Let G be a group, and Y a right (resp. left) G-set. Then we denote by Y/G (resp. $G\backslash Y$) the set of right (resp. left) equivalence classes of Y under G. We denote by $[a]_G$ (resp. by ${}_G[a]$) the right (resp. left) equivalence class of $a \in Y$.

1. SQUARED MÖBIUS FUNCTION FOR HALF-INTEGRAL MATRICES

Throughout the rest of the paper, for two symmetric matrices A and A' with entries in $\mathbf{Q}_p$ we write $A \sim A'$ instead of $A \underset{\mathbf{Z}_p}{\sim} A'$ unless mentioned otherwise. For a non-zero p-adic number c let $v(c) = v_p(c)$ denote the normalized additive valuation of c. For a p-adic number c put

$$\chi_p(c) = 1, \ -1 \text{ or } 0$$

according as $\mathbf{Q}_p(c^{1/2}) = \mathbf{Q}_p$, $\mathbf{Q}_p(c^{1/2})/\mathbf{Q}_p$ is quadratic unramified, or $\mathbf{Q}_p(c^{1/2})/\mathbf{Q}_p$ is quadratic ramified. Furthermore, for a symmetric matrix A of even degree n with entries in $\mathbf{Q}_p$ we put

$$\xi_p(A) = \chi_p((-1)^{n/2} \det A).$$

Here we understand that $\xi_p(A) = 1$ if $n = 0$. Now for a half-integral matrix V over $\mathbf{Z}_p$ let M_V be the quadratic module over $\mathbf{Z}_p$ associated with V, and $\bar{M}_V = M_V \otimes \mathbf{Z}_p/p\mathbf{Z}_p$. A half-integral matrix V is called non-degenerate modulo p if $\bar{M}_V$ is non-degenerate over $\mathbf{Z}_p/p\mathbf{Z}_p$. In the case of $p \neq 2$, V is non-degenerate modulo p if and only if V is unimodular, where as, V is non-degenerate modulo 2 if and only if $V \sim \frac{1}{2}U$ or $V \sim \frac{1}{2}U \perp c_0$ with U an even-integral unimodular matrix and $c_0 \in \mathbf{Z}_2^*$. In general, for a half-integral matrix A over $\mathbf{Z}_p$ we have

$$\bar{M}_A = U \perp \operatorname{Rad} \bar{M}_A$$

with U a non-degenerate quadratic submodule of $\bar{M}_A$. Put $l(A)=\operatorname{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} U$ and $m(A)=\operatorname{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} \operatorname{Rad} \bar{M}_A$. If $l(A)$ is even, put $\bar{\xi}_p(A)=1$ or -1 according as U is hyperbolic space or not. Here we make the convention that $\bar{\xi}_p(A)=1$ if $l(A)=0$, and so on. We remark that $\bar{\xi}_p(A)$ is not necessarily equal to the $\xi_p(A)$. For example, if A is expressed as $A \sim U \perp pV$ with $U \in \frac{1}{2}(\mathscr{E}_{2k}(\mathbf{Z}_p) \cap \Lambda_{2k,\,p})$ and $V \in \frac{1}{2}\mathscr{E}_{n_1}(\mathbf{Z}_p)$, we have $\bar{\xi}_p(A)=\xi_p(U)$.

We define a subset $\mathscr{K}'_n(\mathbf{Z}_p)$ of $\mathscr{H}_n(\mathbf{Z}_p)$ by

$$\mathscr{K}'_n(\mathbf{Z}_p)=\{A \in \mathscr{H}_n(\mathbf{Z}_p);\, A \sim V_0 \perp pV_1 \text{ with } V_0,\, V_1 \text{ non-degenerate matrices modulo } p\}.$$

Furthermore, we define a subset $\mathscr{K}''_n(\mathbf{Z}_2)$ of $\mathscr{H}_n(\mathbf{Z}_2)$ by

$$\mathscr{K}''_n(\mathbf{Z}_2)=\{A \in \mathscr{H}_n(\mathbf{Z}_2);\, A \sim \tfrac{1}{2}V_0 \perp V \perp V_1 \text{ with } V_0,\, V_1 \text{ even-integral unimodular matrices and } V \text{ a diagonal unimodular matrix of degree 2 such that } \det V \equiv 1 \bmod 4\}.$$

We then put $\mathscr{K}_n(\mathbf{Z}_p)=\mathscr{K}'_n(\mathbf{Z}_2) \cup \mathscr{K}''_n(\mathbf{Z}_2)$ or $\mathscr{K}'_n(\mathbf{Z}_p)$ according as $p=2$ or not. If A belongs to $\mathscr{K}_n(\mathbf{Z}_p)$ and $m(A)$ is even, A can be expressed as

$$A \sim V_0 \perp pV_1$$

with V_0 non-degenerate matrix modulo p of degree $n-m(A)$ and $V_1 \in \mathscr{H}_{m(A)}(\mathbf{Z}_p)^{\times}$. Then we put $\tilde{\xi}_p(A)=\xi_p(V_1)$. Now for a non-degenerate half-integral matrix A over $\mathbf{Z}_p$ we define $\sigma_p(A)$ by

$$\sigma_p(A)=\begin{cases}(-1)^{m/2}\,\tilde{\xi}_p(A)\,p^{(m^2-2m)/4} & \text{if } \ m=m(A) \text{ is even}\\ (-1)^{(m-1)/2}\,p^{(m-1)^2/4} & \text{if } \ m=m(A) \text{ is odd.}\end{cases}$$

We explain $\sigma_p(A)$ more precisely; first assume that A belongs to $\mathscr{K}'_n(\mathbf{Z}_p)$. Then we have $A \sim V_0 \perp pV_1$ with V_0, V_1 non-degenerate matrices modulo p of degree n_0 and n_1, respectively. Then we have $m(A)=n_1$ and

$$\sigma_p(A)=\begin{cases}(-1)^{n_1/2}\,\xi_p(V_1)\,p^{(n_1^2-2n_1)/4} & \text{if } n_1 \text{ is even}\\ (-1)^{(n_1-1)/2}\,p^{(n_1-1)^2/4} & \text{if } n_1 \text{ is odd.}\end{cases}$$

Next let $p=2$ and assume that A belongs to $\mathscr{K}''_n(\mathbf{Z}_2)$. Then we have $A \sim \frac{1}{2}V_0 \perp V \perp V_1$ with V_0, V_1 even-integral unimodular matrices of degree n_0 and n_1, respectively, and V a unimodular diagonal matrix of degree 2 such that $\det V \equiv 1 \bmod 4$. In this case, we remark that $m(A) = n_1 + 1$. Thus we have

$$\sigma_p(A) = (-1)^{n_1/2}\, p^{n_1^2/4}.$$

Finally if $A \in \mathscr{H}_n(\mathbf{Z}_p)^\times$ does not belong to $\mathscr{K}_n(\mathbf{Z}_p)$ we put $\sigma_p(A) = 0$. For a non-degenerate half-integral matrix A over $\mathbf{Z}$ put

$$\sigma(A) = \prod_p \sigma_p(A).$$

We note that $\sigma(A)$ is well defined for such a matrix A because we have $\sigma_p(A) = 1$ for almost all p. Define a subset $\mathscr{K}_n(\mathbf{Z})$ of $\mathscr{H}_n(\mathbf{Z})$ by

$$\mathscr{K}_n(\mathbf{Z}) = \{A \in \mathscr{H}_n(\mathbf{Z});\ A \in \mathscr{K}_n(\mathbf{Z}_p) \text{ for any prime number } p\}.$$

Then $\mathscr{K}_n(\mathbf{Z})$ is the support of σ. Moreover by definition $\sigma(A)$ depends only the genus of A. In the case of $n = 1$, the set $\mathscr{H}_1(\mathbf{Z})^\times$ can be identified with the set of non-zero integers, and $\sigma(m) = \mu(m)^2$ for any positive integer m, where μ is the Möbius function. Thus we call σ the squared Möbius function. We should add one remark. Several authors define analogues of the usual Möbius function for abelian groups or for modules (cf. [Ki2], [Sh]). As for the relation between these functions and our squared Möbius function, we discuss in the remark after Theorem 1.

Now for positive definite half-integral matrices A and B of degree m and n, respectively, over $\mathbf{Z}$ we define the representation number $a(A, B)$ of B by A as

$$a(A, B) = \#\{X \in M_{mn}(\mathbf{Z});\ A[X] = B\},$$

and put

$$G(A, B) = \sum_{A' \in \mathscr{G}(A)} \frac{a(A', B)}{a(A', A')},$$

where $\mathscr{G}(A)$ is the set of all Λ_n-equivalence classes belonging to the genus of A. As is well-known $G(A, B)$ depends only on the genera of A and B. Then one of our main results in this paper is:

THEOREM 1. *Let A be a positive definite half-integral matrix of degree n over $\mathbf{Z}$. Then we have*

$$\sum_{\mathscr{G}(A_0) \in \mathscr{G}_n^+} \sigma(A_0)\, G(A_0, A) = 1,$$

where $\mathscr{G}_n^+$ is the set of all genera of positive definite half-integral matrices of degree n.

From the above theorem we immediately obtain

COROLLARY. *Under the same notation and the assumption as above,*

$$\sum_{A_0} \frac{\sigma(A_0)\, a(A_0, A)}{a(A_0, A_0)} = 1,$$

where A_0 runs over all Λ_n-equivalence classes of positive definite half-integral matrices of degree n.

Remark. (1) For indefinite half-integral matrices A and B of m and n, respectively, over $\mathbf{Z}$ "the measure" $\mu(A, B)$ of representation of B by A can be defined (cf. [Si2]), and the above theorem remains valid with a slight modification if we replace $G(A, B)$ by $\mu(A, B)$.

(2) The above corollary can be rewritten as follows:

$$\sum_{D \in \Lambda_n \backslash M_n(\mathbf{Z})^\times} \sigma(A[D^{-1}]) = 1.$$

Now for an element D of $M_n(\mathbf{Z}_p)^\times$, we put $\pi_p(D) = (-1)^i p^{\langle i-1 \rangle}$ or 0 according as D belongs to $\Lambda_{np}(E_{n-i} \perp pE_i)\, \Lambda_{np}$ for some $0 \leqslant i \leqslant n$, or not. Here we write $\langle j \rangle = j(j+1)/2$ for an integer j. Furthermore, for an element D of $M_n(\mathbf{Z})^\times$ put $\pi(D) = \prod_p \pi_p(D)$. This is a certain generalization of the Möbius function (cf. [Ki2]). Let $\mathscr{S}F_n$ denote the set of all elements of $M_n(\mathbf{Z})$ whose elementary divisors are all square free. Then clearly we have $\pi(D) \neq 0$ if and only if D belongs to $\mathscr{S}F_n$. Then by the inversion formula for the generalized Möbius function, for any positive definite half-integral matrix A of degree n, $\sigma(A)$ can be expressed as follows,

$$\sigma(A) = \sum_{D \in \Lambda_n \backslash M_n(\mathbf{Z})^\times} \pi(D)\, \mathbf{1}(A[D^{-1}]),$$

or locally for any prime number p we have

$$\sigma_p(A) = \sum_{D \in \Lambda_{np} \backslash M_n(\mathbf{Z}_p)^\times} \pi_p(D)\, \mathbf{1}_p(A[D^{-1}]),$$

where $\mathbf{1}$ (resp. $\mathbf{1}_p$) denotes the characteristic function of $\mathscr{H}_n(\mathbf{Z})^+$ (resp. $\mathscr{H}_n(\mathbf{Z}_p)^\times$). That is, the recursion formula in the above corollary characterizes σ and σ_p. However, it is not so trivial to derive such explicit forms of σ_p and σ directly from the above expressions.

2. PROOF OF THEOREM 1

In this section, we give a proof of Theorem 1. To prove the theorem, we reduce the problem to the local case, and prove some recursion formula for the local squared Möbius function σ_p (cf. Theorem 2.12). Key ingredients for its proof are an explicit formula for a certain local density (cf. Theorem 2.10) and some combinatorial theoretic identities on quadratic forms over the finite field (cf. Proposition 2.11).

We denote by $(*, *)_p$ the Hilbert symbol on $\mathbf{Q}_p$, and by $h(A)=h_p(A)$ the Hasse invariant of A for a non-degenerate symmetric matrix A with entries in $\mathbf{Q}_p$ (cf. [Ki3]). Let $H=\left(\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right)$, and H_k the orthogonal sum of k-copies of H. From now on, for two elements a and b of $\mathbf{Z}_p \setminus \{0\}$ we write $a \sim b$ if $ab^{-1} \in \mathbf{Z}_p^{*\square}$, and we use the same symbol a to denote the coset of a modulo $\mathbf{Z}_p^{*\square}$. The following two lemmas can be proved by using [Ki3, Theorem 3.4.2].

LEMMA 2.1. *Let n be odd, and $C \in \mathscr{H}_n(\mathbf{Z}_p)$. Assume that we have $C \sim U_0 \perp pU_1$ with U_0 and U_1 non-degenerate matrices modulo p of degree n_0 and n_1, respectively. Then we have*

$$h_p(C)=(-1, -1)_p^{(n^2-1)/8}\,((-1)^{(n+1)/2}, \det U_0 \det U_1)_p$$
$$\times \begin{cases} ((-1)^{(n+1)/2}, p)_p\, \chi_p((-1)^{n_0/2} \det U_0) & n_1 \text{ even} \\ \chi_p((-1)^{n_1/2} \det U_1) & n_1 \text{ even.} \end{cases}$$

LEMMA 2.2. *Let n be even, and $C \in \mathscr{H}_n(\mathbf{Z}_p)$.*

(1) *Let $p \neq 2$, and $C \sim U_0 \perp pU_1$ with U_0 and U_1 unimodular matrices of degree n_0 and n_1, respectively. Then we have*

$$h_p(C)=\begin{cases} \chi_p((-1)^{n_1/2} \det U_1) & n_1 \text{ even} \\ \chi_p((-1)^{(n_1+1)/2} \det U_0) & n_1 \text{ odd.} \end{cases}$$

(2) *Let $p=2$.*

(2.1) *Let $C \sim \frac{1}{2}U_0 \perp U_1$ with U_0 and U_1 even-integral unimodular matrices of degree n_0 and n_1, respectively. Then we have*

$$h_p(C)=(-1)^{n(n+2)/8}\, \chi_p((-1)^{n_1/2} \det U_1).$$

(2.2) *Let $C \sim \frac{1}{2}U_0 \perp U_1 \perp W$, where U_0 and U_1 are even-integral unimodular matrices of degree n_0 and n_1, respectively, and $W=c_0 \perp 2c_1$*

with $c_0, c_1 \in \mathbf{Z}_2^*$ *or* W *is a diagonal unimodular matrix of degree* 2 *such that* $\det W \equiv 1 \bmod 4$. *Then we have*

$$h_p(C) = (-1)^{n(n-2)/8} ((-1)^{n/2-1}, \det W)_2 \, h_2(W).$$

A non-degenerate symmetric matrix A with entries in $\mathbf{Q}_p$ is said to be (a)-maximal if the associated lattice L_A is (a)-maximal in the sense of [Ki3].

Remark. (1) Two (a)-maximal matrices A and A' with entries in $\mathbf{Q}_p$ of the same degree with the same determinant and Hasse invariant are equivalent over $\mathbf{Z}_p$ with each other by [Ki3, Theorem 3.5.2] and [Ki3, Theorem 5.2.2].

(2) Let $c_0, c_1 \in \mathbf{Z}_2^*$. Then $c_0 \perp 2c_1$ is (2)-maximal by [Ki3, Lemma 5.2.1]. Furthermore, if $c_0 c_1 \equiv 1 \bmod 4$, then $c_0 \perp c_1$ is (1)-maximal by [Ki3, Theorem 5.2.1].

Lemma 2.3. *Let* $A, A' \in \mathscr{H}_n(\mathbf{Z}_p)$. *Assume that* $h(A) = h(A')$ *and* $\det A \sim \det A'$. *Then* A *and* A' *are equivalent over* $\mathbf{Z}_p$ *with each other.*

Proof. First assume that $p \neq 2$, or that $p = 2$ and n is odd. Then A and A' can be expressed as

$$A \sim U_0 \perp pU_1, \qquad \text{and} \qquad A' \sim U'_0 \perp pU'_1$$

where U_i and U'_i $(i = 0, 1)$ are non-degenerate matrices of degree n_i and n'_i, respectively. Since we have

$$\det A \sim p^{n_1} \det U_0 \det U_1 \sim p^{n'_1} \det U'_0 \det U'_1 \sim \det A',$$

we have $n_1 = n'_1$, $n_0 = n'_0$ and $\det U_0 \det U_1 \sim \det U'_0 \det U'_1$. Moreover, by Lemmas 2.1 and 2.2 combined with the above equalities we have

$$\chi_p((-1)^{n_1/2} \det U_1) = \chi_p((-1)^{n_1/2} \det U'_1)$$

or

$$\chi_p((-1)^{[n_0/2]} \det U_0) = \chi_p((-1)^{[n_0/2]} \det U'_0)$$

according as n_1 is even or odd. Thus, in any case, we have $\det U'_1 \det U_1^{-1} \in \mathbf{Z}_p^{*\square}$, and $\det U'_0 \det U_0^{-1} \in \mathbf{Z}_p^{*\square}$. Thus we have $U_1 \sim U'_1$ and $U_0 \sim U'_0$. This proves the assertion for the first case.

Next assume that $p=2$ and n is even. Let $n_0=-v_2(\det A)$ and $n_1=n-n_0$. Then by the theory of canonical forms in [W], A can be expressed as one of the following forms:

(a) $A\sim\frac{1}{2}U_0\perp U_1$ with $U_i\in\mathscr{E}_{n_i}(\mathbf{Z}_2)\cap\Lambda_{n_i2}$,

(b) $A\sim H_{(n_0+1)/2}\perp c_0\perp 2H_{(n_1-3)/2}\perp 2c_1$ with $c_i\in\mathbf{Z}_2^*$,

(c) $A\sim H_{n_0/2}\perp V_0\perp 2H_{(n_1-2)/2}$ with V_0 a diagonal unimodular matrix of degree 2 such that $\det V_0\equiv 1 \bmod 4$.

In case (a), by comparing the determinants of A and A', we easily show that A' can be expressed as

$$A'\sim\tfrac{1}{2}U'_0\perp U'_1$$

with $U'_i\in\mathscr{E}_{n_i}(\mathbf{Z}_2)$ and $\det U_0\det U_1\sim\det U'_0\det U'_1$. Then the assertion follows from (2.1) of Lemma 2.2. In case (b), A' can be expressed as

$$A'\sim H_{(n_0+1)/2}\perp c'_0\perp 2H_{(n_1-3)/2}\perp 2c'_1\qquad\text{with}\quad c'_i\in\mathbf{Z}_2^*.$$

Furthermore, (2.2) of Lemma 2.2 shows that the matrices $c_0\perp 2c_1$ and $c'_0\perp 2c'_1$ have the same determinant and Hasse invariant. Thus by the remark before Lemma 2.3, the two matrices are equivalent over $\mathbf{Z}_2$ with each other. Thus the assertion holds for this case. In case (c), A' can be expressed as

$$A'\sim H_{n_0/2}\perp V'_0\perp 2H_{(n_1-2)/2},$$

where V'_0 is a diagonal unimodular matrix of degree 2 such that $\det V'_0\equiv 1 \bmod 4$. In this case, the assertion can also be proved by (2.2) of Lemma 2.2 and the remark before Lemma 2.3.

For an element $A\in S_n(\mathbf{Q}_p)^\times$ let $i(A)$ denote the least integer l such that $p^lA^{-1}\in\mathscr{H}_n(\mathbf{Z}_p)$. Furthermore, put

$$\mathscr{S}F_n(\mathbf{Z}_p)=\bigcup_{i=0}^{n}\Lambda_{np}(E_{n-i}\perp pE_i)\,\Lambda_{np}.$$

Lemma 2.4. *Let $A\in\mathscr{K}_n(\mathbf{Z}_p)$. Let $D\in M_n(\mathbf{Z}_p)^\times$ such that $A[D^{-1}]\in\mathscr{H}_n(\mathbf{Z}_p)$. Then we have $D\in\mathscr{S}F_n(\mathbf{Z}_p)$ and $A[D^{-1}]\in\mathscr{K}_n(\mathbf{Z}_p)$. In particular in the case $p=2$, $A[D^{-1}]$ belongs to $\mathscr{K}'_n(\mathbf{Z}_2)$ or $\mathscr{K}''_n(\mathbf{Z}_2)$ according as A belongs to $\mathscr{K}'_n(\mathbf{Z}_2)$ or $\mathscr{K}''_n(\mathbf{Z}_2)$.*

Proof. Put $A'=A[D^{-1}]$. If $A'\in\mathscr{H}_n(\mathbf{Z}_p)$ and $D\notin\mathscr{S}F_n(\mathbf{Z}_p)$, then A can be expressed as $A\sim\begin{pmatrix}B_{11} & p^2B_{12}\\ p^2B_{21} & p^4B_{22}\end{pmatrix}$ with some $B_{11}\in\mathscr{H}_{n_1}(\mathbf{Z}_p)$, $B_{22}\in\mathscr{H}_{n_2}(\mathbf{Z}_p)$

$(n_1 \geqslant 0, n_2 > 0)$ and $B_{12} = {}^tB_{21} \in \frac{1}{2}M_{n_1 n_2}(\mathbf{Z}_p)$. This implies that we have $A \notin \mathscr{K}_n(\mathbf{Z}_p)$. Thus we have $D \in \mathscr{S}F_n(\mathbf{Z}_p)$. As easily seen, we have $i(A') \leqslant i(A)$, and thus in particular $i(A') \leqslant 1$ for $A \in \mathscr{K}_n(\mathbf{Z}_p)$. Thus the assertion can easily be derived from the uniqueness of Jordan decomposition in the case of $p \neq 2$. Furthermore, we have $\det A' \det A^{-1} \in \mathbf{Q}_p^{\square}$, and in particular $v_p(\det A) \equiv v_p(\det A') \bmod 2$. Thus the assertion can also be derived from the uniqueness of Jordan decomposition except for the case $p = 2$, and $n \equiv v_2(\det A) \equiv 0 \bmod 2$. In the remaining case, A' can be expressed as

$$A' \sim \tfrac{1}{2}U'_0 \perp V' \perp U'_1$$

with U'_0 and U'_1 even-integral unimodular matrices over $\mathbf{Z}_2$ and V' a unimodular diagonal matrix with entries in $\mathbf{Z}_2$ of degree 0 or 2. Now assume that A belongs to $\mathscr{K}'_n(\mathbf{Z}_2)$. Then we have $i(A) \leqslant -1$, and so is $i(A')$. This implies $\deg V' = 0$ and therefore A belongs to $\mathscr{K}'_n(\mathbf{Z}_2)$. Next assume that A belongs to $\mathscr{K}''_n(\mathbf{Z}_2)$. Then we have

$$\begin{aligned} \det U'_0 \det U'_1 \det V' &\equiv p^{-v_2(\det A')} \det A' \\ &\equiv p^{-v_2(\det A)} \det A \equiv (-1)^{(n-2)/2} \bmod 4. \end{aligned}$$

If $\deg V' = 0$ we have

$$\det U'_0 \det U'_1 \equiv (-1)^{n/2} \bmod 4,$$

which contradicts the above equality. Thus we have $\deg V' = 2$ and $\det V' \equiv 1 \bmod 4$. This completes the proof.

Let $A \in \mathscr{K}_n(\mathbf{Z}_p)$. First let $p \neq 2$. Then by the theory of Jordan decomposition A is equivalent to

$$U_0 \perp pH_k \perp pV$$

with U_0 unimodular and V unimodular anisotropic. We then put

$$A^{(i)} = U_0 \perp H_i \perp pH_{k-i} \perp pV.$$

Next let $p = 2$ and $A \in \mathscr{K}'_n(\mathbf{Z}_2)$. Then by the theory of canonical decomposition [W], A is equivalent to

$$U_0 \perp 2H_k \perp 2V$$

with U_0 non-degenerate matrix modulo 2, and V unimodular anisotropic matrix. We then put

$$A^{(i)} = U_0 \perp H_i \perp 2H_{k-i} \perp 2V.$$

Finally let $p=2$ and $A \in \mathscr{K}''_n(\mathbf{Z}_2)$. Then, again by the theory of canonical decomposition [W], A is equivalent to

$$\tfrac{1}{2}U_0 \perp V \perp 2H_k$$

with U_0 even-integral unimodular matrix, and V unimodular matrix of degree 2 such that $\det V \equiv 1 \bmod 4$. Then we put

$$A^{(i)} = \tfrac{1}{2}U_0 \perp H_i \perp V \perp 2H_{k-i}.$$

In any case, $A^{(i)}$ is determined by A and i up to equivalence over $\mathbf{Z}_p$. Now the following proposition follows directly from Lemmas 2.3 and 2.4.

PROPOSITION 2.5. *Let $A \in \mathscr{K}_n(\mathbf{Z}_p)$. Let $D \in M_n(\mathbf{Z}_p)^\times$, and assume that $A[D^{-1}] \in \mathscr{H}_n(\mathbf{Z}_p)$. Then we have $D \in \Lambda_{np}(E_{n-i} \perp pE_i)\,\Lambda_{np}$ with some $0 \leqslant i \leqslant n$, and*

$$A[D^{-1}] \sim A^{(i)}.$$

Now for a non-degenerate half-integral matrix A of degree m over $\mathbf{Z}_p$ and a non-degenerate symmetric matrix B of degree n with entries in $\mathbf{Q}_p$, we define the local density $\alpha_p(A, B)$ and the primitive local density $\alpha_p(A, B)^*$ representing B by A as

$$\alpha_p(A, B) = 2^{-\delta_{mn}} \lim_{e \to \infty} p^{e(-mn+n(n+1)/2)} \# \mathscr{A}_e(A, B),$$

and

$$\alpha_p(A, B)^* = 2^{-\delta_{mn}} \lim_{e \to \infty} p^{e(-mn+n(n+1)/2)} \# \mathscr{A}_e(A, B)^*,$$

where δ_{mn} is Kronecker's delta,

$$\mathscr{A}_e(A, B) = \{X \in M_{mn}(\mathbf{Z}_p)/p^e M_{mn}(\mathbf{Z}_p);\ A[X] - B \in p^e \mathscr{H}_n(\mathbf{Z}_p)\},$$

and

$$\mathscr{A}_e(A, B)^* = \{X \in \mathscr{A}_e(A, B);\ \operatorname{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} X = n\}.$$

Furthermore, put

$$G_p(A, B) = \frac{\alpha_p(A, B)}{\alpha_p(A, A)}\, p^{\nu_p(\det B)(m-n-1)/2 + \nu_p(\det A)(m-n+1)/2}$$

and

$$G_p(A, B)^* = \frac{\alpha_p(A, B)^*}{\alpha_p(A, A)}\, p^{\nu_p(\det B)(m-n-1)/2 + \nu_p(\det A)(m-n+1)/2}.$$

Then by Siegel's main theorem on quadratic forms (cf. [Si1, Satz 1]) we have

$$G(A, B) = \frac{\varepsilon_{m,n}}{\varepsilon_{m,m}} 2^{(m(m-1)-n(n-1))/2} \pi^{mn/2 - m(m+1)/4 - n(n-1)/4}$$

$$\times \frac{\prod_{i=0}^{m-1} \Gamma((m-i)/2)}{\prod_{i=0}^{n-1} \Gamma((m-i)/2)} \prod_p G_p(A, B),$$

for a positive definite half-integral matrices A and B over $\mathbf{Z}$, where $\Gamma(s)$ is Gamma function, and

$$\varepsilon_{m,n} = \begin{cases} 1/2 & \text{if} \quad m = n+1 \quad \text{or} \quad \text{if} \quad m = n > 1 \\ 1 & \text{otherwise.} \end{cases}$$

In particular, if $m = n$, we have

$$G(A, B) = \prod_p G_p(A, B).$$

Now to calculate $G_p(A, B)$ for $A, B \in \mathscr{H}_n(\mathbf{Z}_p)$, we give some preliminary results. For non-negative integers i, j, and a symmetric matrix A of degree n with entries in $\mathbf{Q}_p$ whose scale is included in $\frac{1}{2}\mathbf{Z}_p$, put

$$\mathscr{S}W_{nj}(i, A) = \{X \in M_{nj}(\mathbf{Z}_p)/pM_{nj}(\mathbf{Z}_p);\ A[X] \in p\mathscr{H}_j(\mathbf{Z}_p),\ \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} X = i\}.$$

Put $\bar{\Lambda}_{jp} = GL_j(\mathbf{Z}_p/p\mathbf{Z}_p)$. Clearly $\bar{\Lambda}_{jp}$ naturally acts on the set $\mathscr{S}W_{nj}(i, A)$ by right-multiplication. Thus $\mathscr{S}W_{nj}(i, A)$ can be regarded as a right $\bar{\Lambda}_{jp}$-set. We note that there exists a bijection from $\mathscr{S}W_{nb}(b, A)$ to the $\mathsf{PM}_p(b, n; 2A)$ in [A2, p. 244].

Proposition 2.6. *Let $A \in \mathscr{H}_n(\mathbf{Z}_p)$. Then we have*

$$\#\mathscr{S}W_{nn}(i, A)/\bar{\Lambda}_{np} = \#\mathscr{S}W_{ni}(i, A)/\bar{\Lambda}_{ip}.$$

Proof. For a while, we simply write $[X]_n = [X]_{\bar{\Lambda}_{np}}$, and the others. For an element $[X]_n$ of $\mathscr{S}W_{nn}(i, A)/\bar{\Lambda}_{np}$ there exists an element $U \in \bar{\Lambda}_{np}$ such that $XU = (X_1, O_{n, n-i})$ with $X_1 \in M_{ni}(\mathbf{Z}_p)/pM_{ni}(\mathbf{Z}_p)$. Then we have

$$A[X][U] = \begin{pmatrix} A[X_1] & O \\ O & O \end{pmatrix}.$$

Thus we have $X_1 \in \mathscr{S}W_{ni}(i, A)$. The right $\bar{\Lambda}_{ip}$-equivalence class of X_1 is uniquely determined by $[X]_n$. Thus we can define a mapping Ψ from $\mathscr{S}W_{nn}(i, A)/\bar{\Lambda}_{np}$ to $\mathscr{S}W_{ni}(i, A)/\bar{\Lambda}_{ip}$ by $\Psi([X]_n) = [X_1]_i$. It can easily be shown that Ψ is bijective, and the assertion holds.

For a symmetric matrix A of degree n with entries in $\mathbf{Q}_p$ put

$$\mathscr{F}_n(i, A) = \{X \in \Lambda_{np}(E_{n-i} \perp pE_i)\, \Lambda_{np};\, A[X^{-1}] \in \mathscr{H}_n(\mathbf{Z}_p)\}.$$

PROPOSITION 2.7. (1) *Let $A = \frac{1}{2}U_0 \perp A_1$ with U_0 an even-integral unimodular matrix of degree n_0, and A_1 a symmetric matrix of degree n_1 with entries in $\mathbf{Q}_p$ of the scale $\subset \frac{p}{2}\mathbf{Z}_p$. Put $n = n_0 + n_1$. Then we have*

$$\# \Lambda_{np} \backslash \mathscr{F}_n(i, A) = \# \mathscr{S}W_{n_1 i}\left(i, \frac{1}{p} A_1\right) \Big/ \bar{\Lambda}_{ip}.$$

(2) *Let $p = 2$ and $A = \frac{1}{2}U_0 \perp V \perp A_1$ with U_0 an even-integral unimodular matrix of degree n_0, V a diagonal unimodular matrix of degree n_0', and A_1 an even-integral matrix of degree n_1. Put $n = n_0 + n_0' + n_1$.*

(2.1) *Let $V = (c)$. Then we have*

$$\# \Lambda_{n2} \backslash \mathscr{F}_n(i, A) = \# \mathscr{S}W_{n_1 i}(i, \tfrac{1}{2}A_1)/\bar{\Lambda}_{i2}.$$

(2.2) *Let $V = c_1 \perp c_2$ with $c_1, c_2 \in \mathbf{Z}_2^*$. Then we have*

$$\# \Lambda_{n2} \backslash \mathscr{F}_n(i, A) = \# \mathscr{S}W_{n_1+1, i}(i, \tfrac{1}{2}A_1 \perp \tfrac{1}{2}(c_1 + c_2))/\bar{\Lambda}_{i2}.$$

Proof. (1) Let $X \in \mathscr{F}_n(i, A)$. Then without loss of generality we may assume that

$$X = \begin{pmatrix} X_{00} & X_{01} \\ O & X_{11} \end{pmatrix}$$

with $X_{rr} \in M_{n_r}(\mathbf{Z}_p)^\times$ $(r = 0, 1)$ and $X_{01} \in M_{n_0 n_1}(\mathbf{Z}_p)$. By assumption $A[X^{-1}]$ belongs to $\mathscr{H}_n(\mathbf{Z}_p)$, and thus we have

$$\tfrac{1}{2}U_0[X_{00}^{-1}] \in \mathscr{H}_{n_0}(\mathbf{Z}_p),\ \tfrac{1}{2}U_0[X_{00}^{-1}]\, X_{01} X_{11}^{-1} \in \tfrac{1}{2}M_{n_0 n_1}(\mathbf{Z}_p),$$

and

$$U_0[X_{00}^{-1}][X_{01}X_{11}^{-1}]+A_1[X_{11}^{-1}]\in\mathscr{H}_{n_1}(\mathbf{Z}_p).$$

Since U_0 is even-integral unimodular, X_{00} is unimodular, and X is right Λ_{np}-equivalent to $\begin{pmatrix} E_{n_0} & O \\ O & X_{11}\end{pmatrix}$. This implies $\operatorname{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} pX_{11}^{-1}=i$. Furthermore, clearly we have $\frac{1}{p}A_1[pX_{11}^{-1}]\in p\mathscr{H}_{n_1}(\mathbf{Z}_p)$, that is $pX_{11}^{-1}\in\mathscr{S}W_{n_1n_1}(i,\frac{1}{p}A_1)$. The right Λ_{n_1p}-equivalence class of pX_{11}^{-1} depends only on ${}_n[X]$. Thus we can define a mapping Φ from $\Lambda_{np}\backslash\mathscr{F}_n(i,A)$ to $\mathscr{S}W_{n_1n_1}(i,\frac{1}{p}A_1)/\Lambda_{n_1p}$ by $\Phi({}_n[X])=[pX_{11}^{-1}]_{n_1}$. This is bijective. Thus the assertion follows from Proposition 2.6.

(2) By (1) we have

$$\#\Lambda_{n2}\backslash\mathscr{F}_n(i,A)=\#\mathscr{S}W_{n_1+n_0',\,i}(i,\tfrac{1}{2}V\perp\tfrac{1}{2}A_1)/\bar{\Lambda}_{i2}.$$

Put $X=\begin{pmatrix}\mathbf{x}_1\\ X_2\end{pmatrix}$ with $\mathbf{x}_1\in M_{1i}(\mathbf{Z}_2)/2M_{1i}(\mathbf{Z}_2)$ and $X_2\in M_{n_1i}(\mathbf{Z}_2)/2M_{n_1i}(\mathbf{Z}_2)$. In the case of (2.1), X belongs to $\mathscr{S}W_{n_1+1,\,i}(i,\frac{1}{2}c\perp\frac{1}{2}A_1)$ if and only if $\mathbf{x}_1=\mathbf{0}$ and $X_2\in\mathscr{S}W_{n_1i}(i,\frac{1}{2}A_1)$. Thus the projection Pr from $M_{n_1+1,\,i}(\mathbf{Z}_2)/2M_{n_1+1,\,i}(\mathbf{Z}_2)$ onto $M_{n_1i}(\mathbf{Z}_2)/2M_{n_1i}(\mathbf{Z}_2)$ defined by $\begin{pmatrix}\mathbf{x}_1\\ X_2\end{pmatrix}\to X_2$ induces a bijection from $\mathscr{S}W_{n_1+1,\,i}(i,\frac{1}{2}c\perp\frac{1}{2}A_1)/\bar{\Lambda}_{i2}$ to $\mathscr{S}W_{n_1i}(i,\frac{1}{2}A_1)/\bar{\Lambda}_{i2}$, and therefore the assertion holds. In the case of (2.2), if X belongs to $\mathscr{S}W_{n_1+2,\,i}(i,\frac{1}{2}c_1\perp\frac{1}{2}c_2\perp\frac{1}{2}A_1)$, we have $\mathbf{x}_1=\mathbf{x}_2$, and $X_2\in\mathscr{S}W_{n_1+1,\,i}(i,\frac{1}{2}(c_1+c_2)\perp\frac{1}{2}A_1)$, where $\mathbf{x}_2$ denotes the first row of X_2. Conversely, if X_2 belongs to $\mathscr{S}W_{n_1+1,\,i}(i,\frac{1}{2}(c_1+c_2)\perp\frac{1}{2}A_1)$, we have $\begin{pmatrix}\mathbf{x}_2\\ X_2\end{pmatrix}\in\mathscr{S}W_{n_1+2,\,i}(i,\frac{1}{2}c_1\perp\frac{1}{2}c_2\perp\frac{1}{2}A_1)$, where $\mathbf{x}_2$ denotes the first row of X_2. Thus the projection Pr also induces a bijection from $\mathscr{S}W_{n_1+2,\,i}(i,\frac{1}{2}c_1\perp\frac{1}{2}c_2\perp\frac{1}{2}A_1)$ to $\mathscr{S}W_{n_1+1,\,i}(i,\frac{1}{2}(c_1+c_2)\perp\frac{1}{2}A_1)$, which completes the assertion.

COROLLARY. *Let $p=2$.*

(1) *Let $V=c_1\perp c_2$ with $c_1c_2\equiv 1 \bmod 4$. Then we have*

$$\#\Lambda_{n2}\backslash\mathscr{F}_n(i,A)=\#\mathscr{S}W_{n_1+1,\,i}(i,\tfrac{1}{2}A_1\perp 1)/\bar{\Lambda}_{i2}.$$

(2) *Let $V=c_1\perp c_2$ with $c_1c_2\equiv 3 \bmod 4$. Then we have*

$$\#\Lambda_{n2}\backslash\mathscr{F}_n(i,A)=\#\mathscr{S}W_{n_1+1,\,i}(i,\tfrac{1}{2}A_1\perp 2)/\bar{\Lambda}_{i2}.$$

For a non-degenerate matrix A modulo p of degree n put

$$J(i,A,p)=\begin{cases}(p^{n/2}-\xi_p(A))(p^{n/2-i}+\xi_p(A))\prod_{j=1}^{i-1}(p^{n-2j}-1) & n \text{ is even}\\ \prod_{j=1}^{i}(p^{n-2j+1}-1) & n \text{ is odd.}\end{cases}$$

Here we understand that $J(i, A, p)=1$ if $i=0$. Furthermore, put $\phi_i(x)=\prod_{j=1}^{i}(x^j-1)$. We note that for a symmetric matrix A of degree m with entries in $\mathbf{Q}_p$ we have

$$\#(\mathscr{S}W_{mi}(i, A)/\bar{\Lambda}_{ip})=\frac{\#\mathscr{S}W_{mi}(i, A)}{\#\bar{\Lambda}_{ip}}$$

and

$$\#\bar{\Lambda}_{ip}=p^{\langle i-1\rangle}\phi_i(p).$$

Thus the following proposition follows from Proposition 2.7 and [A2, Proposition A.2.14].

Proposition 2.8. *Let $A\in\mathscr{K}_n(\mathbf{Z}_p)$.*

(1) *Let $p\neq 2$, or $p=2$ and $A\in\mathscr{K}'_n(\mathbf{Z}_p)$. Let $A\sim U_0\perp pU_1$ with U_0 and U_1 non-degenerate modulo p. Then we have*

$$\#\Lambda_{np}\backslash\mathscr{F}_n(i, A)=\frac{J(i, U_1, p)}{\phi_i(p)}.$$

(2) *Let $p=2$ and $A\in\mathscr{K}''_n(\mathbf{Z}_2)$. Let $A\sim\frac{1}{2}U_0\perp V\perp U_1$ with U_0 and U_1 even-integral unimodular matrices and V a diagonal unimodular matrix of degree 2 such that* $\det V\equiv 1 \bmod 4$. *Then we have*

$$\#\Lambda_{n2}\backslash\mathscr{F}_n(i, A)=\frac{J(i, \frac{1}{2}U_1\perp 1, 2)}{\phi_i(2)}.$$

The following lemma is useful for treating several combinatorial problems in this paper.

Lemma 2.9. *Let x and t be variables, and m be a positive integer. Then we have*

(1) $$\sum_{i=0}^{m} t^i\frac{x^{\langle i\rangle}\phi_m(x)}{\phi_i(x)\,\phi_{m-i}(x)}=\prod_{\alpha=1}^{m}(1+tx^{\alpha}).$$

(2) $$\sum_{i=0}^{m} x^{i^2-mi}t^i\prod_{j=1}^{m-i}(1-x^{1-j}t)\frac{\phi_m(x)}{\phi_i(x)\,\phi_{m-i}(x)}=1.$$

Proof. (1) is nothing but [A2, (3.2.34))], and (2) can be proved by replacing q with x^{-1} in the formula of [I-S, Lemma 5.5].

COROLLARY. *Let x be a variable. For a positive integer m and $\chi = \pm 1$ put*

$$F(m, \chi, x) = \sum_{i=0}^{m} (-1)^{m-i} x^{(m-i)^2-(m-i)} \times \frac{(x^m-\chi)(x^{m-i}+\chi) \prod_{j=1}^{i-1} (x^{2m-2j}-1)}{\phi_i(x)},$$

and

$$G(m, x) = \sum_{i=0}^{m} (-1)^{m-i} x^{(m-i)^2} \frac{\prod_{j=1}^{i} (x^{2m+2-2j}-1)}{\phi_i(x)}.$$

Then we have $F(m, \chi, x) = G(m, x) = 1$ for any m and χ.

Proof. First by a simple calculation we have

$$G(m, x) = \sum_{i=0}^{m} (-1)^i x^{i^2} \frac{\phi_m(x)}{\phi_i(x)\, \phi_{m-i}(x)} \prod_{j=1}^{m-i} (1 + x^{m+1-j}),$$

and

$$F(m, 1, x) = \sum_{i=0}^{m} (-1)^i x^{i^2-i} \frac{\phi_m(x)}{\phi_i(x)\, \phi_{m-i}(x)} \prod_{j=1}^{m-i} (1 + x^{m-j}),$$

Thus the assertion for these two cases can be proved by replacing t with $-x^m$ and $-x^{m-1}$, respectively, in (2) of Lemma 2.9. Finally we have

$$F(m, -1, x) = -F(m, 1, x) + 2G(m-1, x).$$

Thus the assertion also holds for this case.

Let A and B be half-integral matrices over an integral domain R of degree m, and n, respectively. We say A represents B over R if there exists a square matrix X with entries in R such that $A[X] = B$. In particular if $m = n$, we say A dominates B over R. Let $\mathcal{H}_n(R)^\times / GL_n(R)$ denote the set of $GL_n(R)$-equivalence classes of non-degenerate half-integral symmetric matrices of degree n. For an element $A \in \mathcal{H}_n(R)^\times$, we denote by $[A]$ the equivalence class of A in $\mathscr{H}_n(R)^\times/GL_n(R)$. For two elements $[A], [B] \in \mathscr{H}_n(R)^\times/GL_n(R)$ we write $[A] \leqslant [B]$ if A dominates B over R. This definition does not depend on the choice of the representatives of the classes, and by this relation "$\leqslant$" we can define the order on the set $\mathscr{H}_n(R)/GL_n(R)$. In particular we write $[A] < [B]$ if $[A] \leqslant [B]$ and $[A] \neq [B]$. We often write $A < B$ instead of $[A] < [B]$. We note that an element $[A]$ in $\mathscr{H}_n(\mathbf{Z}_p)/\Lambda_{np}$ is minimal with respect to the order "$\leqslant$" if and only if A is (1)-maximal in the sense of Section 1. Let A and B be non-degenerate half-integral matrices over $\mathbf{Z}_p$. Then we note that A represents B over $\mathbf{Z}_p$ if and only if $G_p(A, B) \neq 0$.

THEOREM 2.10. *Let* $A, B \in \mathscr{H}_n(\mathbf{Z}_p)$, *and* $i = (v_p(\det B) - v_p(\det A))/2$. *Assume that A dominates B over* $\mathbf{Z}_p$.

(1) *Let $B \sim U_0 \perp pU_1$ with U_0, U_1 non-degenerate modulo p. Then we have*

$$G_p(A, B) = \frac{J(i, U_1, p)}{\phi_i(p)}.$$

(2) *Let $p = 2$ and $B \sim \frac{1}{2}U_0 \perp V \perp U_1$ with U_0, U_1 even-integral unimodular and V a diagonal unimodular matrix of degree 2 such that* $\det V \equiv 1 \bmod 4$. *Then we have*

$$G_2(A, B) = \frac{J(i, \frac{1}{2}U_1 \perp 1, 2)}{\phi_i(2)}.$$

Proof. By Proposition 2.5 we have $A \sim B^{(i)}$. By [Ki3, Theorem 5.6.1] we have

$$G_p(A, B) = \sum_{D \in \Lambda_{np} \backslash M_n(\mathbf{Z}_p)^\times} G_p(A, B[D^{-1}])^*.$$

Again by Proposition 2.5, we have $G_p(A, B[D^{-1}])^* = 1$ or 0 according as $D \in \mathscr{F}_n(i, B)$ or not. Thus we have

$$G_p(A, B) = \# \Lambda_{np} \backslash \mathscr{F}_n(i, B).$$

This proves the assertion by Proposition 2.8.

To complete the proof of Theorem 1, we give several relations among $\mathscr{S}W_{ni}(i, A)$'s.

PROPOSITION 2.11. *Let A be a non-degenerate half-integral matrix over $\mathbf{Z}_p$.*

(1) *Let $A \notin \mathscr{K}_n(\mathbf{Z}_p)$. Then we have*

$$\sum_{i=0}^{n} (-1)^i p^{\langle i-1\rangle} \#(\Lambda_{np} \backslash \mathscr{F}_n(i, A)) = 0.$$

(2) *Let $A \in \mathscr{K}_n(\mathbf{Z}_p)$. Then we have*

$$\sum_{A_0} \sigma_p(A_0)\, G_p(A_0, A) = 1,$$

where A_0 runs over all Λ_{np}-equivalence classes of elements of $\mathscr{K}_n(\mathbf{Z}_p)$.

Proof. (1) Let $A \notin \mathscr{K}_n(\mathbf{Z}_p)$. Then by Proposition 2.7, there is a positive integer l and a half-integral matrix $\tilde{A}$ of degree $l-1$ such that

$$\#(\Lambda_{np} \backslash \mathscr{F}_n(i, A)) = \frac{\# \mathscr{S} W_{li}(i, 0 \perp \tilde{A})}{\# \bar{\Lambda}_{ip}} = \frac{\# \mathscr{S} W_{li}(i, 0 \perp \tilde{A})}{p^{\langle i-1\rangle} \phi_i(p)}\ .$$

On the other hand, by a simple calculation we have

$$\# \mathscr{S} W_{li}(i, 0 \perp \tilde{A}) = p^i \# \mathscr{S} W_{l-1,\, i}(i, \tilde{A}) + \frac{\phi_i(p)\, p^{\langle i-1\rangle} \# \mathscr{S} W_{l-1,\, i-1}(i-1, \tilde{A})}{\phi_{i-1}(p)\, p^{\langle i-2\rangle}}.$$

Thus we have

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i p^{\langle i-1\rangle} \#(\Lambda_{np} \backslash \mathscr{F}_n(i, A)) \\
&= \sum_{i=0}^{l} (-1)^i p^{\langle i-1\rangle} \frac{\# \mathscr{S} W_{li}(i, 0 \perp \tilde{A})}{p^{\langle i-1\rangle} \phi_i(p)} \\
&= \sum_{i=0}^{l} (-1)^i p^{\langle i-1\rangle} \left\{ \frac{p^i \# \mathscr{S} W_{l-1,\, i}(i, \tilde{A})}{\phi_i(p)\, p^{\langle i-1\rangle}} + \frac{\# \mathscr{S} W_{l-1,\, i-1}(i-1, \tilde{A})}{\phi_{i-1}(p)\, p^{\langle i-2\rangle}} \right\} \\
&= \sum_{i=0}^{l} (-1)^i p^i \frac{\# \mathscr{S} W_{l-1,\, i}(i, \tilde{A})}{\phi_i(p)} \\
&\quad - \sum_{i=0}^{l} (-1)^{i-1} p^{i-1} \frac{\# \mathscr{S} W_{l-1,\, i-1}(i-1, \tilde{A})}{\phi_{i-1}(p)} = 0.
\end{aligned}$$

This proves the assertion.

(2) First let $A \sim U_0 \perp pU_1$ with U_0, U_1 non-degenerate modulo p of degree n_0 and n_1, respectively. By Proposition 2.8,

$$\sum_{A_0} \sigma_p(A_0)\, G_p(A_0, A) = F(n_1/2, \xi_p(U_1), p) \text{ or } G((n_1-1)/2, p)$$

according as n_1 is even or odd, where $F(n_1/2, \xi_p(U_1), x)$ and $G((n_1-1)/2, x)$ are the polynomials in Corollary to Lemma 2.9. Thus the assertion follows from the corollary. Similarly the assertion holds for $A \in \mathscr{K}''_n(\mathbf{Z}_2)$.

THEOREM 2.12. *Let $A \in \mathscr{H}_n(\mathbf{Z}_p)^\times$. Then we have*

$$\sum_{A_0} \sigma_p(A_0)\, G_p(A_0, A) = 1,$$

where A_0 runs over all Λ_{np}-equivalence classes of elements of $\mathscr{K}_n(\mathbf{Z}_p)$.

Proof. We prove the assertion by induction with respect to the order "$\leqslant$". The assertion clearly holds if A is minimal with respect to "$\leqslant$". Assume that A is not minimal and that the assertion holds for any A' such that $A' < A$. If $A \in \mathscr{K}_n(\mathbf{Z}_p)$, the assertion holds by (2) of Proposition 2.11. If $A \notin \mathscr{K}_n(\mathbf{Z}_p)$, by [Ki1, Theorem 1] we have

$$G_p(A_0, A) = \sum_{i=1}^{n} (-1)^{i-1} p^{\langle i-1 \rangle} \sum_{D \in \Lambda_{np} \backslash \Lambda_{np}(E_{n-i} \perp pE_i)\Lambda_{np}} G_p(A_0, A[D^{-1}])$$

for any $A_0 \in \mathscr{K}_n(\mathbf{Z}_p)$. Thus we have

$$\begin{aligned}\sum_{A_0} \sigma_p(A_0)\, G_p(A_0, A) &= \sum_{i=1}^{n} (-1)^{i-1} p^{\langle i-1 \rangle} \sum_{D \in \Lambda_{np} \backslash \Lambda_{np}(E_{n-i} \perp pE_i)\Lambda_{np}} \sum_{A_0} \sigma_p(A_0) \\ &\quad \times G_p(A_0, A[D^{-1}]).\end{aligned}$$

By the induction hypothesis we have

$$\sum_{A_0} \sigma_p(A_0)\, G_p(A_0, A[D^{-1}]) = 1$$

for any $i \geqslant 1$ and $D \in \Lambda_{np} \backslash \Lambda_{np}(E_{n-i} \perp pE_i)\Lambda_{np}$ such that $A[D^{-1}] \in \mathscr{H}_n(\mathbf{Z}_p)$. Thus by (1) of Proposition 2.11 we have

$$\sum_{A_0} \sigma_p(A_0)\, G_p(A_0, A) = \sum_{i=1}^{n} (-1)^{i-1} p^{\langle i-1 \rangle} \#(\Lambda_{np} \backslash \mathscr{F}(i, A)) = 1.$$

This proves the assertion.

Proof of Theorem 1. By Siegel's main theorem on quadratic forms (cf. [Si1, Satz 1]), for any positive definite half-integral matrix A over $\mathbf{Z}$ of degree n we have

$$\sum_{\mathscr{G}(A_0)} \sigma(A_0)\, G(A_0, A) = \frac{1}{2}\Bigg(\prod_p \sum_{A_0'} \sigma_p(A_0')\, G_p(A_0', A) + \prod_p \sum_{A_0'} h_p(A_0')\, \sigma_p(A_0')\, G_p(A_0', A)\Bigg),$$

where $\mathscr{G}(A_0)$ runs over all genera of positive definite half-integral matrices of degree n, and A_0' runs over all Λ_{np}-equivalence classes of non-degenerate half-integral matrices over $\mathbf{Z}_p$ of degree n (cf. [I-S]). We note that if $G_p(A_0', A) \neq 0$, then we have $h_p(A_0') = h_p(A)$. We also note that we have $\prod_p h_p(A) = 1$. Thus we have

$$\begin{aligned}\prod_p \sum_{A_0'} h_p(A_0')\, \sigma_p(A_0')\, G_p(A_0', A) &= \prod_p h_p(A) \sum_{A_0'} \sigma_p(A_0')\, G_p(A_0', A)\\ &= \prod_p \sum_{A_0'} \sigma_p(A_0')\, G_p(A_0', A).\end{aligned}$$

Thus the assertion follows from Theorem 2.12.

3. KOECHER–MAAß DIRICHLET SERIES FOR SIEGEL MODULAR FORM

In this section, using Theorem 1, we obtain a reasonable expression of the Koecher–Maaß Dirichlet series for a Siegel modular form. Put

$$GSp_n(\mathbf{Q}) = \{M \in GL_{2n}(\mathbf{Q});\, J_n[M] = \kappa(M)\, J_n \text{ with some } \kappa(M) > 0\},$$

and

$$Sp_n(\mathbf{Z}) = \{M \in \Lambda_{2n};\, J_n[M] = J_n\},$$

where $J_n = \left(\begin{smallmatrix} O_n & E_n \\ -E_n & O_n \end{smallmatrix}\right)$, and let $\mathbf{L}_n = \mathbf{L}(GSp_n(\mathbf{Q}), Sp_n(\mathbf{Z}))$ be the Hecke algebra associated with the pair $(GSp_n(\mathbf{Q}), Sp_n(\mathbf{Z}))$. Let $\mathbf{H}_n$ be the Siegel's upper half-space. A function f on $\mathbf{H}_n$ is called a modular form of weight k belonging to $Sp_n(\mathbf{Z})$, or simply a Siegel modular form of degree n, if it satisfies the following conditions:

(i) f is holomorphic on $\mathbf{H}_n$;

(ii) $f((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k f(Z)$ for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbf{Z})$;

(iii) if $n=1$, for any $\alpha > 0$, $f(z)$ is bounded on each set $\{x+iy;\ y \geqslant \alpha\}$.

Let f be a modular form of weight k belonging to $Sp_n(\mathbf{Z})$. Let

$$f(Z) = \sum_A a_f(A)\, exp(2\pi i\, tr(AZ))$$

be the Fourier expansion of $f(Z)$, where A runs over all semi-positive definite half-integral matrices of degree n and tr denotes the trace of a matrix. We then define the Koecher–Maaß Dirichlet series $L(f, s)$ for f by

$$L(f, s) = \sum_A \frac{a_f(A)}{a(A, A)(\det A)^s},$$

where A runs over all Λ_n-equivalence classes of positive definite half-integral matrices of degree n.

Now we denote by $M_k(Sp_n(\mathbf{Z}))$ the module of modular forms of weight k belonging to $Sp_n(\mathbf{Z})$. Then, for any element g of $\mathbf{L}_n$, we can define an action of g on $M_k(Sp_n(\mathbf{Z}))$ in a usual way. We call this action the Hecke operator as usual (cf. [A2].) Put

$$GSp_n(\mathbf{Q}_p) = \{M \in GL_{2n}(\mathbf{Q}_p);\ J_n[M] = \kappa(M)\, J_n \text{ with some } \kappa(M) \in \mathbf{Q}_p^\times\},$$

and

$$Sp_n(\mathbf{Z}_p) = \{M \in \Lambda_{2n,\, p};\ J_n[M] = J_n\},$$

and let $\mathbf{L}_{np} = \mathbf{L}(GSp_n(\mathbf{Q}_p), Sp_n(\mathbf{Z}_p))$ be the Hecke algebra associated with the pair $(GSp_n(\mathbf{Q}_p), Sp_n(\mathbf{Z}_p))$. We now review the Satake parameters of $\mathbf{L}_{np}$; let $\mathbf{P}_n = \mathbf{C}[X_0^{\pm}, X_1^{\pm}, ..., X_n^{\pm}]$ be the ring of Laurent polynomials in $X_0, X_1, ..., X_n$ over $\mathbf{C}$. Let $\mathbf{W}_n$ be the group of $\mathbf{C}$-automorphisms of $\mathbf{P}_n$ generated by all permutations in variables $X_1, ..., X_n$ and by the auto morphisms $\tau_1, ..., \tau_n$ defined by

$$\tau_i(X_0) = X_0 X_i,\ \tau_i(X_i) = X_i^{-1},\ \tau_i(X_j) = X_j\ (j \neq i]).$$

Furthermore, a group $\widetilde{\mathbf{W}}_n$ isomorphic to $\mathbf{W}_n$ acts on the set $\mathbf{T}_n = (\mathbf{C}^\times)^{n+1}$ in a way similarly to above. Then there exists a $\mathbf{C}$-algebra isomorphism Φ_{np}, called the Satake isomorphism, from $\mathbf{L}_{np}$ to the $\mathbf{W}_n$-invariant subring

$\mathbf{P}_n^{\mathbf{W}_n}$ of $\mathbf{P}_n$. Then for a $\mathbf{C}$-algebra homomorphism λ from $\mathbf{L}_{np}$ to $\mathbf{C}$, there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), ..., \alpha_n(p, \lambda))$ of $\mathbf{T}_n$ satisfying

$$\lambda(\Phi_{np}^{-1}(F(X_0, X_1, ..., X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), ..., \alpha_n(p, \lambda))$$

for $F \in \mathbf{P}_n^{\mathbf{W}_n}$. The equivalence class of $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), ..., \alpha_n(p, \lambda))$ under the action of $\tilde{\mathbf{W}}_n$ is uniquely determined by λ. We call this the Satake parameters of $\mathbf{L}_{np}$ determined by λ. Now assume that an element f of $M_k(Sp_n(\mathbf{Z}))$ is a common eigenfunction of all Hecke operators. Then for each prime number p, f defines a $\mathbf{C}$-algebra homomorphism $\lambda_{f,p}$ from $\mathbf{L}_{np}$ to $\mathbf{C}$ in a usual way, and hence we obtain the Satake parameters $(\alpha_0(p, \lambda_{f,p}), \alpha_1(p, \lambda_{f,p}), ..., \alpha_n(p, \lambda_{f,p}))$. We then define the standard L-function $\zeta^+(f, s)$ by

$$\zeta^+(f, s) = \prod_p \prod_{i=1}^{n} \{(1 - \alpha_i(p) p^{-s})(1 - \alpha_i(p)^{-1} p^{-s})\}^{-1},$$

where $\alpha_i(p) = \alpha_i(p, \lambda_{f,p})$ $(i = 1, ..., n)$. Now for a half-integral matrix A over $\mathbf{Z}_p$ let $l = l(A)$ and $\bar{\xi}_p(A)$ be those defined in Section 1. Then we define Andrianov's polynomial $B_p(v, A)$ as follows:

$$B_p(v, A) = \begin{cases} (1+v)(1 - \bar{\xi}_p(A) p^{-l/2} v) \prod_{i=1}^{l/2-1} (1 - p^{-2i} v^2) & l \text{ is even} \\ (1+v) \prod_{i=1}^{(l-1)/2} (1 - p^{-2i} v^2) & l \text{ is odd.} \end{cases}$$

Here we understand that we have $B_p(v, A) = 1$ if $l = 0$. We note that this is nothing but the polynomial $B_p^n(v, 2A)$ in [A2, Proposition 4.2.24] if A is half-integral over $\mathbf{Z}$. We should remark that Andrianov defined the polynomial $B_p^n(v, A)$ for an even-integral matrix A over $\mathbf{Z}$ in [A2], but it can also be defined for a half-integral matrix over $\mathbf{Z}_p$ as above with a slight modification.

Now let A be a positive definite half-integral matrix over $\mathbf{Z}$. We define

$$X(s, f, A) = \sum_{D \in \Lambda_n \backslash \mathcal{S}F_n} \pi(D)(\det D)^{-s-n+k} a_f(A[D^{-1}])$$

for a positive definite half-integral matrix A over $\mathbf{Z}$. We note that this essentially coincides with the one in [A2, (4.3.81)]. We also remark that $X(s, f, A) = a_f(A)$ if A is maximal in $\mathcal{H}_n(\mathbf{Z})^+$, and that $X(k-n, f, A)$ coincides with the *primitive Fourier coefficient* $a_f^*(A)$ which will be defined below. Here a half-integral matrix A over $\mathbf{Z}$ is called maximal in $\mathcal{H}_n(\mathbf{Z})^+$

if there is no non-unimodular square matrix X with entries in $\mathbf{Z}$ such that $A[X^{-1}] \in \mathscr{H}_n(\mathbf{Z})^+$. Put

$$B(s, A) = \prod_p B_p(p^{-s}, A),$$

$$Y(s, f, A) = \sum_{A' \in \mathscr{G}(A)} \frac{X(s, f, A')}{a(A', A')},$$

and

$$K(f, s) = \sum_{\mathscr{G}(A_0) \in \mathscr{G}_n^+} \frac{\sigma(A_0)\, B(2s-k+1, A_0)\, Y(2s-k+1, f, A_0)}{(\det A_0)^s}.$$

Then we have

PROPOSITION 3.1. *Let $f \in M_k(Sp_n(\mathbf{Z}))$ be a common eigenfunction of all Hecke operators. Then we have*

$$L(f, s) = \zeta^+(f, 2s-k+1)\, K(f, s).$$

Remark. The above theorem for $n=1$ is nothing but the equality in the Introduction.

Proof. By [A2, Theorem 4.3.19], for any A of $\mathscr{H}_n(\mathbf{Z})^+$ we have

$$\sum_{M \in M_n(\mathbf{Z})^\times / \Lambda_n} \frac{a_f(A[M])}{(\det M)^{2s}}$$
$$= B(2s-k+1, A)\, X(2s-k+1, f, A)\, \zeta^+(f, 2s-k+1)$$

(see also [B1], [Sh]). Thus we have

$$\begin{aligned} &K(f, s)\, \zeta^+(f, 2s-k+1) \\ &= \sum_{\mathscr{G}(A_0) \in \mathscr{G}_n^+} \frac{\sigma(A_0)}{(\det A_0)^s} \sum_{A' \in \mathscr{G}(A_0)} \sum_{M \in M_n(\mathbf{Z})^\times / \Lambda_n} \frac{a_f(A'[M])}{(\det M)^{2s}\, a(A', A')} \\ &= \sum_{\mathscr{G}(A_0) \in \mathscr{G}_n^+} \sigma(A_0) \sum_{A' \in \mathscr{G}(A_0)} \sum_{A} \frac{a(A', A)\, a_f(A)}{a(A, A)\, a(A', A')(\det A)^s} \\ &= \sum_{A} \frac{a_f(A)}{a(A, A)(\det A)^s} \sum_{\mathscr{G}(A_0) \in \mathscr{G}_n^+} \sigma(A_0) \sum_{A' \in \mathscr{G}(A_0)} \frac{a(A', A)}{a(A', A')}, \end{aligned}$$

where A runs over all Λ_n-equivalence classes of positive half-integral matrices of degree n. By Theorem 1, for any A, we have

$$\sum_{\mathscr{G}(A_0)\in\mathscr{G}_n^+} \sigma(A_0) \sum_{A'\in\mathscr{G}(A_0)} \frac{a(A',A)}{a(A',A')}=1.$$

Thus the assertion holds.

We remark that by definition we have $\sigma(A)=0$ for $A\notin\mathscr{K}_n(\mathbf{Z})$. Thus to compute $K(f,s)$ it suffices to treat $X(s,f,A)$ only for $A\in\mathscr{K}_n(\mathbf{Z})$. Thus $K(f,s)$ is much easier to treat than $L(f,s)$ itself.

Now to give a more reasonable formula of $K(f,s)$ we rewrite $Y(s,f,A_0)$ in more concise form. For $A\in\mathscr{H}_n(\mathbf{Z})^+$, put

$$M(A)=\sum_{A'\in\mathscr{G}(A)} \frac{1}{a(A',A')}.$$

Now let A and B be non-degenerate half-integral matrices over $\mathbf{Z}_p$. If A dominates B over $\mathbf{Z}_p$, there exists a square matrix D with entries in $\mathbf{Z}_p$ such that $B=A[D]$. In particular if $B\in\mathscr{K}_n(\mathbf{Z}_p)$, then by Proposition 2.5, D belongs to $\Lambda_{np}(E_{n-i}\perp pE_i)\,\Lambda_{np}$ for some i. Then we put $\pi_p(A,B)=\pi_p(D)$. We also put $\pi_p(A,B)=0$ if A does not dominate B over $\mathbf{Z}_p$ or if B does not belong to $\mathscr{K}_n(\mathbf{Z}_p)$. By Proposition 2.5, $\pi_p(A,B)$ is uniquely determined by A and B. Finally for positive definite half-integral matrices A and B of degree n over $\mathbf{Z}$, we define $\pi(A,B)=\prod_p \pi_p(A,B)$. We also put $\mathscr{D}_n(m)=\{X\in M_n(\mathbf{Z});\ \det X=m\}$. The following lemma can easily be proved.

LEMMA 3.2. *Let F be a class invariant function on $\mathscr{H}_n(\mathbf{Z})^+$. Then for $B\in\mathscr{H}_n(\mathbf{Z})^+$ we have*

$$\sum_{D\in\Lambda_n\backslash\mathscr{D}_n(m)} F(B[D^{-1}])=\sum_C \frac{a(C,B)\,F(C)}{a(C,C)},$$

where C runs over all Λ_n-equivalence classes of positive definite half-integral matrices of degree n and of determinant $m^{-2}\det B$. In particular, if $B\in\mathscr{K}_n(\mathbf{Z})$, we have

$$\sum_{D\in\Lambda_n\backslash(\mathscr{D}_n(m)\cap\mathscr{S}F_n)} F(B[D^{-1}])=\sum_C \frac{a(C,B)\,F(C)}{a(C,C)},$$

where C runs over all Λ_n-equivalence classes of positive definite half-integral matrices of degree n and of determinant $m^{-2}\det B$.

COROLLARY. *Under the above notation and the assumption we have*

$$\sum_{D \in \Lambda_n \backslash M_n(\mathbf{Z})^\times} F(B[D^{-1}]) = \sum_C \frac{a(C, B)\, F(C)}{a(C, C)},$$

where C runs over all Λ_n-equivalence classes of positive definite half-integral matrices of degree n. In particular, if $B \in \mathscr{K}_n(\mathbf{Z})$, we have

$$\sum_{D \in \Lambda_n \backslash \mathscr{S}F_n} F(B[D^{-1}]) = \sum_C \frac{a(C, B)\, F(C)}{a(C, C)},$$

where C runs over all Λ_n-equivalence classes of positive half-integral matrices of degree n.

Now for a modular form f of weight k belonging to $Sp_n(\mathbf{Z})$ and $A \in \mathscr{H}_n(\mathbf{Z})^+$, following [B-R], we define the A-th *primitive Fourier coefficient* $a_f^*(A)$ of f by

$$a_f^*(A) = \sum_{D \in \Lambda_n \backslash \mathscr{S}F_n} \pi(D)\, a_f(A[D^{-1}]),$$

and put

$$G_f^*(A) = \sum_{C \in \mathscr{G}(A)} \frac{a_f^*(C)}{a(C, C)}.$$

We note that by the inversion formula for the generalized Möbius function we have

$$a_f(A) = \sum_{D \in \Lambda_n \backslash M_n(\mathbf{Z})^\times} a_f^*(A[D^{-1}]).$$

By Corollary to Lemma 3.2, we have

$$a_f(A) = \sum_B \frac{a(B, A)\, a_f^*(B)}{a(B, B)},$$

where B runs over all Λ_n-equivalence classes of positive definite half-integral matrices of degree n. For two matrices A, B of $\mathscr{H}_n(\mathbf{Z})^+$ and a prime number p put $m_p = m_p(A, B) = 1/2(\nu_p(\det B) - \nu_p(\det A))$ or 0 according as A dominates B over $\mathbf{Z}_p$ or not, and put

$$T(s, A, B) = \prod_p \prod_{i=1}^{m_p} (1 - p^{-s-n+i}).$$

THEOREM 3.3. *Let f be as in Proposition* 3.1. *Then for any $A \in \mathcal{K}_n(\mathbf{Z})^+$ we have*

$$Y(s, f, A) = M(A) \sum_{\mathcal{G}(C_0) \in \mathcal{G}_n^+} \frac{G(C_0, A)\, G_f^*(C_0)}{M(C_0)}\, T(s-k+1, C_0, A).$$

Proof. For a while, unless mentioned otherwise, $\sum_B$ means that B runs over all A_n-equivalence classes of positive definite half-integral matrices of degree n. By Lemma 3.2 we have

$$X(s, f, A) = \sum_B \pi(B, A)(\det(AB^{-1}))^{(-s-n+k)/2} \frac{a(B, A)\, a_f(B)}{a(B, B)}.$$

Thus we have

$$\begin{aligned} Y(s, f, A) &= \sum_{A' \in \mathcal{G}(A)} \sum_B \pi(B, A)(\det(AB^{-1}))^{(-s-n+k)/2} \frac{a(B, A')\, a_f(B)}{a(A', A')\, a(B, B)} \\ &= \sum_B \frac{\pi(B, A)\, a_f(B)(\det(AB^{-1}))^{(-s-n+k)/2}}{a(B, B)} \sum_{A' \in \mathcal{G}(A)} \frac{a(B, A')}{a(A', A')}. \end{aligned}$$

By Andrianov's duality theorem in [A1], we have

$$\sum_{A' \in \mathcal{G}(A)} \frac{a(B, A')}{a(A', A')} = \frac{M(A)\, G(B, A)}{M(B)}.$$

Thus we have

$$\begin{aligned} Y(s, f, A) &= \sum_B \frac{\pi(B, A)\, a_f(B)(\det(AB^{-1}))^{(-s-n+k)/2}}{a(B, B)} \frac{M(A)\, G(B, A)}{M(B)} \\ &= \sum_{\mathcal{G}(B_0) \in \mathcal{G}_n^+} \frac{\pi(B_0, A)(\det(AB_0^{-1}))^{(-s-n+k)/2}\, M(A)\, G(B_0, A)}{M(B_0)} \\ &\quad \times \sum_{B' \in \mathcal{G}(B_0)} \frac{a_f(B')}{a(B', B')}. \end{aligned}$$

On the other hand, we have

$$a_f(B') = \sum_C \frac{a(C, B')\, a_f^*(C)}{a(C, C)}.$$

Thus again, by Andrianov's duality theorem, we have

$$\sum_{B' \in \mathscr{G}(B_0)} \frac{1}{a(B', B')} \sum_{C} \frac{a(C, B')\, a_f^*(C)}{a(C, C)}$$
$$= \sum_{C} \frac{a_f^*(C)}{a(C, C)} \sum_{B' \in \mathscr{G}(B_0)} \frac{a(C, B')}{a(B', B')}$$
$$= \sum_{C} \frac{a_f^*(C)}{a(C, C)} \frac{M(B_0)\, G(C, B_0)}{M(C)}.$$

Thus we have

$$Y(s, f, A) = \sum_{\mathscr{G}(B_0) \in \mathscr{G}_n^+} \pi(B_0, A)(\det(AB_0^{-1}))^{(-s-n+k)/2} M(A)$$
$$\times \frac{G(B_0, A)}{M(B_0)} \sum_{C} \frac{a_f^*(C)}{a(C, C)} \frac{M(B_0)\, G(C, B_0)}{M(C)}$$
$$= M(A) \sum_{\mathscr{G}(C_0) \in \mathscr{G}_n^+} \frac{G_f^*(C_0)}{M(C_0)} \sum_{\mathscr{G}(B_0) \in \mathscr{G}_n^+} \pi(B_0, A)$$
$$\times (\det(AB_0^{-1}))^{(-s-n+k)/2}\, G(C_0, B_0)\, G(B_0, A).$$

By Theorem 2.10 we have

$$G(C_0, B_0)\, G(B_0, A) = G(C_0, A) \prod_{p} \frac{\phi_{m_p}(p)}{\phi_{i_p}(p)\, \phi_{m_p - i_p}(p)},$$

where $m_p = m_p(C_0, A)$ and $i_p = m_p(B_0, A)$. Thus we have

$$\sum_{\mathscr{G}(B_0) \in \mathscr{G}_n^+} \pi(B_0, A)(\det(AB_0^{-1}))^{(-s-n+k)/2}\, G(C_0, B_0)\, G(B_0, A)$$
$$= G(C_0, A) \prod_{p} \sum_{i_p=0}^{m_p} (-1)^{i_p} p^{\langle i_p - 1 \rangle} p^{i_p(-s-n+k)} \frac{\phi_{m_p}(p)}{\phi_{i_p}(p)\, \phi_{m_p - i_p}(p)}.$$

Thus the assertion holds by (1) of Lemma 2.9.

Let A and C be elements of $\mathscr{H}_n(\mathbf{Z})^+$. We note that $\sigma(A) \neq 0$ if and only if $A \in \mathscr{K}_n(\mathbf{Z})$. We also note that $C \in \mathscr{K}_n(\mathbf{Z})$ if $A \in \mathscr{K}_n(\mathbf{Z})$ and $G(C, A) \neq 0$. Then by rewriting $K(f, s)$ and using Theorem 3.3 we have the following:

THEOREM 3.4. *Let f be as in Proposition* 3.1. *Then we have*

$$L(f, s) = \zeta^{+}(f, 2s-k+1) \sum_{\mathscr{G}(C_0)} \frac{G_f^*(C_0)}{M(C_0)} \times \sum_{\mathscr{G}(A_0)} \frac{\sigma(A_0)\, M(A_0)\, B(2s-k+1, A_0)}{(\det A_0)^s} \times T(2s-2k+2, C_0, A_0)\, G(C_0, A_0),$$

where $\mathscr{G}(C_0)$ and $\mathscr{G}(A_0)$ run over all genera of $\mathscr{H}_n(\mathbf{Z})^{+}$.

By Proposition 2.8, an explicit form of $G(A, B)$ is given for $A, B \in \mathscr{H}_n(\mathbf{Z})^{+}$. Moreover, as will be seen in Lemma 4.2, an exact value of $M(A)$ is also given. Thus if we obtain an explicit form of $G_f^*(C_0)$, we will know a lot of information on $L(f, s)$. In fact, in the case where f is the Klingen–Eisenstein series, by [B-R] or [Ki2], we know an explicit form of $G_f^*(C_0)$, and therefore give an explicit form of $L(f, s)$ by the above theorem (cf. [I-K2], [Ka]). We also remark that we have given an explicit form of $L(f, s)$ for the Siegel–Eisenstein series f by a different method from this paper (cf. [I-K1]), and that Saito has given a generalization of this result from the view point of the zeta function of prehomogeneous vector space (cf. [Sa]).

4. APPLICATION TO THE ZETA FUNCTION FOR HALF-INTEGRAL MATRICES

We define the Dirichlet series $\zeta_n(s)$ by

$$\zeta_n(s) = \sum_{S} \frac{1}{(\det S)^s\, a(S, S)},$$

where S runs over all Λ_n-equivalence classes of positive definite half-integral matrices of degree n. Note that $\zeta_n(s)$ coincides with $\frac{1}{2}\zeta_n(s, L_n^*)$ in [I-S], and is one of the most important Dirichlet series both in quadratic forms and in modular forms. We further put

$$\zeta(GL_n, s) = \sum_{X \in M_n(\mathbf{Z})^{\times}/\Lambda_n} \frac{1}{|\det X|^s},$$

and

$$K_n(s) = \sum_{\mathscr{G}(A_0) \in \mathscr{G}_n^{+}} \frac{\sigma(A_0)\, M(A_0)}{(\det A_0)^s}.$$

PROPOSITION 4.1. *We have*

$$\zeta(GL_n, s) = \prod_{i=1}^{n} \zeta(s-i+1),$$

and

$$\zeta_n(s) = K_n(s)\, \zeta(GL_n, 2s),$$

where $\zeta(s)$ is the Riemann zeta function.

Proof. The first assertion is well known (cf. [A2, Exercise 3.2.10]). The second can be proved similarly to Proposition 3.1 by using Theorem 1.

Let $n=1$. Then $\zeta_1(s)$ is nothing but half of the Riemann zeta function and thus it has the following expression:

$$\zeta_1(s) = \frac{1}{2}\, \zeta(2s) \sum_{m=1}^{\infty} \frac{\mu(m)^2}{m^s}.$$

This expression is well-known. From now on assume that $n \geqslant 2$. Then the above proposition enables us to deal with $\zeta_n(s)$ very easily. In fact, for a Λ_{np}-invariant function ω_p and $d \in \mathbf{Z}_p \backslash \{0\}$ put

$$K_{n,\,p}(t, d_0, \omega_p) = \sum_{r=0}^{\infty} \sum_{A} \frac{\omega_p(A)\, \sigma_p(A)}{\alpha_p(A, A)}\, (p^{(n+1)/2} t)^{(2r - 2[n/2]\, \delta_{2p} + \nu_p(d_0))},$$

where A runs over all Λ_{np}-equivalence classes of half-integral matrices of degree n over $\mathbf{Z}_p$ such that $\det A = p^{2r - 2[n/2]\, \delta_{2p}}\, d_0$. (The right-hand side of the formula on page 242, line 6 in [Ka] should be read as above.) Now let ι_p be the constant function taking the value 1, and h_p the Hasse invariant. Then by Siegel's main theorem on quadratic forms (cf. [Si1, Satz 1]) combined with the method in [I-S], we have

$$K_n(s) = c_n \sum_{d_0 \in \mathscr{D}_n} \left(\prod_{p} K_{n,\,p}(p^{-s}, d_0, \iota_p) + \prod_{p} K_{n,\,p}(p^{-s}, d_0, h_p) \right),$$

where

$$c_n = \frac{2^{n(n-1)/2} \prod_{i=1}^{n} \Gamma(i/2)}{\pi^{n(n+1)/4}},$$

and

$$\mathscr{D}_n = \{d \in \mathbf{Z}^+;\ (-1)^{n/2}\, d \text{ is the fundamental discriminant of a quadratic field or } 1\},$$

or

$$\mathscr{D}_n = \{d \in \mathbf{Z}^+;\ d \text{ is square free}\},$$

according as n is even or odd. (The constant c_n on page 242, line 16 in [Ka] should be read as above.) We note that $K_{n,\,p}(t, d_0, \omega)$ is a polynomial in t because we have $\sigma_p(A) \neq 0$ for finitely many A (up to Λ_{np}-equivalence) of half-integral matrices over $\mathbf{Z}_p$. Thus we have only to compute some polynomials (not power series) to give an explicit form of $K_n(s)$.

Now we compute $K_{n,\,p}(t, d_0, \omega_p)$. Let $p \neq 2$. Let B be an element of $\mathscr{K}_n(\mathbf{Z}_p)$. Then B is equivalent to the following form,

$$U_0 \perp pU_1,$$

where U_0 and U_1 are symmetric unimodular matrices of degree n_0 and n_1, respectively. Here we understand that U_0 or U_1 is the *empty matrix* according as $n_0 = 0$ or $n_1 = 0$. We say that B is type (i, j) if $(n_0, n_1) \equiv (i, j) \bmod 2$. Moreover put $\Xi^{(0)}(B) = (-1)^{n_0/2} \det U_0$ and $\xi^{(0)}(B) = \xi_p^{(0)}(B) = \chi_p((-1)^{n_0/2} \det U_0)$ if B is type $(0, 0)$ or $(0, 1)$, and $\Xi^{(1)}(B) = (-1)^{n_1/2} \det U_1$ and $\xi^{(1)}(B) = \xi_p^{(1)}(B) = \chi_p((-1)^{n_1/2} \det U_1)$ if B is type $(0, 0)$ or $(1, 0)$. Here we understand that we have $\Xi^{(i)}(B) = 1$ and $\xi^{(i)}(B) = 1$ if $n_i = 0$. Then $\Xi^{(i)}(B)$ is uniquely determined, up to $\mathbf{Z}_p^{*\square}$, by B, and thus $\xi^{(i)}(B)$ is uniquely determined by B.

Let B be an element of $\mathscr{K}_n(\mathbf{Z}_2)$. Then B is exactly one of the following types:

$$
\begin{array}{ll}
(0, 0) & \tfrac{1}{2} U_0 \perp U_1, \\
(1, 1) & H_{n_0/2} \perp c_0 \perp 2H_{n_1/2} \perp 2c_1, \\
(2, 0) & H_{n_0/2} \perp V \perp 2H_{n_1/2}, \\
(1, 0) & H_{n_0/2} \perp c_0 \perp U_1, \\
(0, 1) & \tfrac{1}{2} U_0 \perp 2H_{n_1} \perp 2c_1,
\end{array}
$$

where U_0 and U_1 are even-integral unimodular matrices of degree n_0 and n_1, respectively, V is a diagonal unimodular matrix of degree 2 whose determinant is congruent to 1 modulo 4, and c_0, c_1 are 2-adic units. Here we understand that U_0 is the *empty matrix* if $n_0 = 0$, and the others. Put

$\Xi^{(0)}(B)=(-1)^{n_0/2}\det U_0$ and $\xi^{(0)}(B)=\xi_2^{(0)}(B)=\chi_2((-1)^{n_0/2}\det U_0)$ if B is type $(0,0)$ or $(0,1)$, and $\Xi^{(1)}(B)=(-1)^{n_1/2}\det U_1$ and $\xi^{(1)}(B)=\xi_2^{(1)}(B)=\xi_2((-1)^{n_1/2}\det U_1)$ if B is type $(0,0)$ or $(1,0)$. Furthermore, put $\xi^{(1)}(B)=\xi_2^{(1)}(B)=h_2(c_0\perp 2c_1)$ or $h_2(V)$ according as B is type $(1,1)$ or $(2,0)$. We make the convention that we have $\Xi^{(i)}(B)=1$ and $\xi^{(i)}(B)=1$ if $n_i=0$. The quantity $\Xi^{(i)}(B)$ is uniquely determined, up to $\mathbf{Z}_2^{*\square}$, by B, and $\xi^{(i)}(B)$ is uniquely determined by B.

We remark that $\xi_p^{(0)}(B)=\bar{\xi}_p(B)$ if B is type $(0,0)$ or $(0,1)$, and $\xi_p^{(1)}(B)=\tilde{\xi}_p(B)$ if B is type $(0,0)$ or $(1,0)$, where $\bar{\xi}_p(B)$ and $\tilde{\xi}_p(B)$ are those defined in Section 1. We define a polynomial $\varphi_m(x)$ in x by $\varphi_i(x)=\prod_{j=1}^{i}(1-x^j)$. We note that $\varphi_i(x)=(-1)^i\phi_i(x)$. The following lemma is well known (cf. [Ki3, Theorem 5.6.3]).

LEMMA 4.2. *Let A be an element of $\mathscr{H}_n(\mathbf{Z}_p)$. Let $l=[n/2]\,\delta_{2p}+[v_p(\det A)/2]$.*

(1) *Let $p\neq 2$. Then we have*

$$\alpha_p(A,A)=\begin{cases}\dfrac{2p^{l(2l+1)}\varphi_l(p^{-2})\,\varphi_{n/2-l}(p^{-2})}{(1+p^{-l}\xi^{(1)}(A))(1+p^{-n/2+l}\xi^{(0)}(A))} & \text{if } A \text{ is type } (0,0)\\ 2p^{(l+1)(2l+1)}\varphi_l(p^{-2})\,\varphi_{n/2-l-1}(p^{-2}) & \text{if } A \text{ is type } (1,1)\\ \dfrac{2p^{l(2l+1)}\varphi_l(p^{-2})\,\varphi_{(n-1)/2-l}(p^{-2})}{1+p^{-l}\xi^{(1)}(A)} & \text{if } A \text{ is type } (1,0)\\ \dfrac{2p^{(l+1)(2l+1)}\varphi_l(p^{-2})\,\varphi_{(n-1)/2-l}(p^{-2})}{1+p^{-(n-1)/2+l}\xi^{(0)}(A)} & \text{if } A \text{ is type } (0,1).\end{cases}$$

(2) *Let $p=2$. Then we have*

$$\alpha_2(A,A)=\begin{cases}\dfrac{2^{l(2l+1)+1}\varphi_l(2^{-2})\,\varphi_{n/2-l}(2^{-2})}{(1+2^{-l}\xi^{(1)}(A))(1+2^{-n/2+l}\xi^{(0)}(A))} & A \text{ is type } (0,0)\\ 2^{l(2l-1)+3}\varphi_{l-1}(2^{-2})\,\varphi_{n/2-l}(2^{-2}) & A \text{ is type } (1,1)\\ 2^{l(2l-1)+2}\varphi_{l-1}(2^{-2})\,\varphi_{n/2-l}(2^{-2}) & A \text{ is type } (2,0)\\ \dfrac{2^{l(2l+1)+2}\varphi_l(2^{-2})\,\varphi_{(n-1)/2-l}(2^{-2})}{1+2^{-l}\xi^{(1)}(A)} & A \text{ is type } (1,0)\\ \dfrac{2^{(l+1)(2l+1)+2}\varphi_l(2^{-2})\,\varphi_{(n-1)/2-l}(2^{-2})}{1+2^{-(n-1)/2+l}\xi^{(0)}(A)} & A \text{ is type } (0,1).\end{cases}$$

We denote by $\mathscr{U}=\mathscr{U}_p$ the set $\{1, 5\}$ or a complete set of representatives of $\mathbf{Z}_p^*/\mathbf{Z}_p^{*\square}$ according as $p=2$ or not. Furthermore, for a Λ_{np}-invariant function ω_p and $d_0 \in \mathbf{Z}_p^*$ put

$$H_{n,p}(t, d_0, \omega_p) = K_{n,p}(t, d_0, \omega_p) + K_{n,p}(t, pd_0, \omega_p).$$

PROPOSITION 4.3. *Let n be odd and $d_{0p} \in \mathbf{Z}_p^*$.*

(1) *Let $\omega_p = \iota_p$. Then for $l=0, 1$, $K_{n,p}(t, p^l d_{0p}, \iota_p)$ depends only on l and we have*

$$K_{n,p}(t, p^l d_{0p}, \iota_p) = \frac{(2^{(-1-n^2)/2} t^{-n+1})^{\delta_{2p}} (p^{(n-1)/2} t)^l}{\varphi_{(n-1)/2}(p^{-2})} \prod_{i=1}^{(n-1)/2} (1 - p^{-2i+n-1} t^2).$$

Furthermore, we have

$$H_{n,p}(t, d_{0p}, \iota_p) = \frac{(2^{(-1-n^2)/2} t^{-n+1})^{\delta_{2p}} (1 + p^{(n-1)/2} t)}{\varphi_{(n-1)/2}(p^{-2})} \times \prod_{i=1}^{(n-1)/2} (1 - p^{-2i+n-1} t^2).$$

(2) *Let $\omega_p = h_p$. Then for $l=0, 1$, $((-1)^{(n+1)/2}, d_{0p})_p K_{n,p}(t, p^l d_{0p}, h_p)$ depends only on l and we have*

$$K_{n,p}(t, p^l d_{0p}, h_p) = ((-1)^{(n^2-1)/8} 2^{(-1-n^2)/2} t^{-n+1})^{\delta_{2p}} ((-1)^{(n+1)/2}, d_{0p})_p \times \frac{(((-1)^{(n+1)/2}, p)_p t)^l}{\varphi_{(n-1)/2}(p^{-2})} \prod_{i=1}^{(n-1)/2} (1 - p^{-2i+n} t^2).$$

Furthermore, we have

$$H_{n,p}(t((-1)^{(n+1)/2}, p)_p, d_{0p}, h_p) = ((-1)^{(n^2-1)/8} 2^{(-1-n^2)/2} t^{-n+1})^{\delta_{2p}} ((-1)^{(n+1)/2}, d_{0p})_p \times \frac{1+t}{\varphi_{(n-1)/2}(p^{-2})} \prod_{i=1}^{(n-1)/2} (1 - p^{-2i+n} t^2).$$

Proof. First we note that $\sigma_p(A) \neq 0$ if and only if $A \in \mathscr{K}_n(\mathbf{Z}_p)$. For an element A of $\mathscr{K}_n(\mathbf{Z}_p)$ put $d = \det A$ and $\Xi = \Xi^0(A)$ or $\Xi^{(1)}(A)$ according as A is type $(0, 1)$ or $(1, 0)$. We note that Λ_{np}-equivalence class of A is uniquely determined by d and Ξ by Lemmas 2.1 and 2.3. Thus we denote by $A(d; \Xi)$ its representative. We also note that $A \notin \mathcal{K}_n(\mathbf{Z}_p)$ unless $v_p(\det A) \leqslant n-2[n/2]\,\delta_{2p}$. Thus, by definition, for $i=0, 1$ and $l=0, 1$ we have

$$K_{n,p}(t, p^l d_{0p}, h_p^i)$$
$$= \sum_{\Xi \in \mathcal{U}} \sum_{j=0}^{(n-1)/2} \frac{\sigma_p(A(p^{2j+l-(n-1)\,\delta_{2p}} d_{0p}; \Xi))}{\alpha_p(A(p^{2j+l-(n-1)\,\delta_{2p}} d_{0p}; \Xi), A(p^{2j+l-(n-1)\,\delta_{2p}} d_{0p}; \Xi))}$$
$$\times (p^{(n+1)/2} t)^{2j+l-(n-1)\,\delta_{2p}} h_p(A(p^{2j+l-(n-1)\,\delta_{2p}} d_{0p}; \Xi))^i.$$

Hence, by Lemmas 2.1 and 4.2 we have

$$K_{n,p}(t, d_{0p}, h_p^i) = (2^{-(n+1)/2} t^{-1})^{(n-1)\,\delta_{2p}}\, 2^{-1-\delta_{2p}}$$
$$\times (-1)^{i(n^2-1)\,\delta_{2p}/8} ((-1)^{(n+1)/2}, d_{0p})_p^i$$
$$\times \sum_{\xi=\pm 1} \sum_{j=0}^{(n-1)/2} \frac{(-1)^j p^{-j^2-j+nj} t^{2j}}{\varphi_j(p^{-2})\, \varphi_{(n-1)/2-j}(p^{-2})}\, \xi^{i+1} (1+p^{-j}\xi),$$

and

$$K_{n,p}(t, p\, d_{0p}, h_p^i) = (2^{-(n+1)/2} t^{-1})^{(n-1)\,\delta_{2p}}\, 2^{-1-\delta_{2p}}$$
$$\times (-1)^{i(n^2-1)\,\delta_{2p}/8} ((-1)^{(n+1)/2}, d_{0p})_p^i$$
$$\times p^{(n-1)/2} t \sum_{\xi=\pm 1} \sum_{j=0}^{(n-1)/2} \frac{(-1)^j p^{-j^2-2j+nj} t^{2j}}{\varphi_j(p^{-2})\, \varphi_{(n-1)/2-j}(p^{-2})}$$
$$\times (((-1)^{(n+1)/2}, p)_p\, \xi)^i\, (1+p^{-(n-1)/2+j}\xi).$$

Thus by (1) of Lemma 2.9 we have

$$K_{n,p}(t, d_{0p}, \iota_p) = \frac{(2^{(-1-n^2)/2} t^{-n+1})^{\delta_{2p}}}{\varphi_{(n-1)/2}(p^{-2})} \prod_{i=1}^{(n-1)/2} (1-p^{-2i+n-1} t^2),$$

$$K_{n,p}(t, p\, d_{0p}, \iota_p) = \frac{(2^{(-1-n^2)/2} t^{-n+1})^{\delta_{2p}}\, p^{(n-1)/2} t}{\varphi_{(n-1)/2}(p^{-2})} \prod_{i=1}^{(n-1)/2} (1-p^{-2i+n-1} t^2),$$

$$K_{n,p}(t, d_{0p}, h_p) = \frac{((-1)^{(n^2-1)/8}\, 2^{(-1-n^2)/2} t^{-n+1})^{\delta_{2p}} ((-1)^{(n+1)/2}, d_{0p})_p}{\varphi_{(n-1)/2}(p^{-2})}$$
$$\times \prod_{i=1}^{(n-1)/2} (1-p^{-2i+n} t^2),$$

and

$$K_{n,\,p}(t,\,p\,d_{0p},\,h_p)=\frac{((-1)^{(n^2-1)/8}\,2^{(-1-n^2)/2}t^{-n+1})^{\delta_{2p}}\,((-1)^{(n+1)/2},\,d_{0p})_p}{\varphi_{(n-1)/2}(p^{-2})}$$

$$\times((-1)^{(n+1)/2},\,p)_p\,t\prod_{i=1}^{(n-1)/2}(1-p^{-2i+n}t^2).$$

Thus the assertion holds.

Let n be an even positive integer. For an odd prime number p put

$$\mathscr{D}_{np}=\{d\in\mathbf{Z}_p;\,d=p^l d' \text{ with } l=0 \text{ or } 1, \text{ and } d'\in\mathbf{Z}_p^*\},$$

and

$$\mathscr{D}_{n2}=\{d\in\mathbf{Z}_2;\,(-1)^{n/2}\,d\equiv 1 \bmod 4,\ (-1)^{n/2}\,4^{-1}d\equiv 3 \bmod 4, \text{ or } v_2(d)=3\}.$$

Then by the same argument as above we have

PROPOSITION 4.4. *Let n be even, and $d_0\in\mathscr{D}_{np}$. Put $l_0=v_p(d_0)$.*

(1) *Let $\omega_p=\iota_p$. Then we have*

$$K_{n,\,p}(t,\,d_0,\,\iota_p)=\frac{(2^{-(n+n^2)/2}t^{-n})^{\delta_{2p}}\,(1-\chi_p(d_0)\,p^{n/2-1}t^2)}{(1-\chi_p(d_0)\,p^{-n/2})\,\varphi_{n/2-1}(p^{-2})}$$

$$\times p^{(n-1)\,l_0/2}t^{l_0}\prod_{i=1}^{n/2-1}(1-p^{-2i+n-1}t^2).$$

(2) *Let $\omega_p=h_p$.*

(2.1). *Assume that $l_0=0$. Then we have*

$$K_{n,\,p}(t,\,d_0,\,h_p)=\frac{((-1)^{(n^2+2n)/8}\,\chi_p(d_0)\,2^{-(n+n^2)/2}t^{-n})^{\delta_{2p}}}{(1-p^{-n/2}\chi_p(d_0))\,\varphi_{n/2-1}(p^{-2})}\prod_{i=1}^{n/2}(1-p^{-2i+n}t^2).$$

(2.2). *Assume that $l_0=1+2\delta_{2p}$ or that $p=2$ and $l_0=2$. Then we have*

$$K_{n,\,p}(t,\,d_0,\,h_p)=0.$$

Now let n be an even positive integer. For an element d of $\mathscr{D}_n$ let χ_d be the quadratic Dirichlet character corresponding to the extension $\mathbf{Q}(((-1)^{n/2}d)^{1/2})/\mathbf{Q}$, and $L(s, \chi_d)$ the Dirichlet L-series associated with χ_d. Here we understand $\chi_d = 1$ if $(-1)^{n/2}d = 1$. We then define a Dirichlet series $D_n^*(s)$ by

$$D_n^*(s) = (-1)^{[n/4]} \sum_{d \in \mathscr{D}_n} 2(2\pi)^{-n/2}(n/2-1)!\, d^{-s+(n-1)/2} L(n/2, \chi_d) \times \frac{\zeta(2s)\,\zeta(2s-n+1)}{L(2s-n/2+1, \chi_d)}.$$

This Dirichlet series was introduced by Cohen in [C]. Moreover let B_j be the jth Bernoulli number. Then we can essentially reprove [I-S, Theorems 1.1 and 1.2] for the positive definite case:

THEOREM 4.5. (1) *Let n be an odd integer $\geqslant 3$. Then we have*

$$\zeta_n(s) = \frac{\left|\prod_{i=1}^{(n-1)/2} B_{2i}\right|}{2^n \left(\dfrac{n-1}{2}\right)!} 2^{(n-1)s} \left\{ \zeta(s-(n-1)/2) \prod_{i=1}^{(n-1)/2} \zeta(2s-(2i-1)) + (-1)^{(n^2-1)/8} \zeta(s) \prod_{i=1}^{(n-1)/2} \zeta(2s-2i) \right\}.$$

(2) *Let n be an even integer $\geqslant 2$. Then we have*

$$\zeta_n(s) = \frac{\left|\prod_{i=1}^{n/2-1} B_{2i}\right|}{2^n \left(\dfrac{n-2}{2}\right)!} 2^{ns} \left\{ (-1)^{[n/4]} D_n^*(s) \prod_{i=1}^{n/2-1} \zeta(2s-2i) + (1+(-1)^{n/2})(-1)^{(n^2+n)/8} \frac{|B_{n/2}|}{n} \prod_{i=1}^{n/2} \zeta(2s-(2i-1)) \right\}.$$

Proof. For a family $\Omega = \{\omega_p\}_p$ of functions on $H_n(\mathbf{Z}_p)^\times$ put

$$K_n(s, \Omega) = \sum_{d_0 \in \mathscr{D}_n} \prod_p K_{n,p}(p^{-s}, d_0, \omega_p).$$

First let n be odd. Then by Proposition 4.3 and using the same argument as in the proof of [I-S, Theorems 1.1] we can prove

$$K_n(s, \{\iota_p\}_p) = 2^{(-1-n^2)/2} 2^{s(n-1)} \prod_{i=1}^{(n-1)/2} \zeta(2i) \times \frac{\zeta(s-(n-1)/2)}{\prod_{i=0}^{(n-1)/2} \zeta(2s+2i-n+1)},$$

and

$$K_n(s, \{h_p\}_p) = (-1)^{(n^2-1)/8}\, 2^{(-1-n^2)/2} 2^{s(n-1)} \prod_{i=1}^{(n-1)/2} \zeta(2i) \times \frac{\zeta(s)}{\zeta(2s) \prod_{i=1}^{(n-1)/2} \zeta(2s+2i-n)}.$$

Thus the assertion follows from Proposition 4.1.

Next let n be even. Then by Proposition 4.4 and using the same argument as in the proof of [I-S, Theorems 1.2] we can prove

$$K_n(s, \{\iota_p\}_p) = \frac{2^{(-n-n^2)/2} 2^{sn} \prod_{i=1}^{n/2-1} \zeta(2i)}{\zeta(2s) \prod_{i=0}^{n/2-1} \zeta(2s+2i-n+1)} \times \sum_{d_0 \in \mathscr{D}_n} \frac{d_0^{-s+(n-1)/2} L(n/2, \chi_{d_0})}{L(2s-n/2+1, \chi_{d_0})}.$$

Finally we calculate $K_n(s, \{h_p\}_p)$. If $d_0 \in \mathscr{D}_n$ is different from 1, by (2.2) of Proposition 4.4, we have

$$\prod_p K_{n,p}(p^{-s}, d_0, h_p) = 0.$$

Thus, if $n \equiv 2 \bmod 4$, we have

$$K_n(s, \{h_p\}_p) = 0.$$

On the other hand, if $n \equiv 0 \bmod 4$, we have

$$K_n(s, \{h_p\}_p) = \prod_p K_{n,p}(p^{-s}, 1, h_p)$$

$$= \frac{2^{(-n-n^2)/2} 2^{sn} (-1)^{(n^2+2n)/8} \zeta(n/2) \prod_{i=1}^{n/2-1} \zeta(2i)}{\prod_{i=1}^{n/2} \zeta(2s+2i-n)}.$$

Thus the assertion follows again from Proposition 4.1.

Remark. We can also prove the theorem on the zeta functions for indefinite half-integral symmetric matrices in the same way.

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